Homework due. Wednesday 2/5/20 via upload to Canvas.

Reading material. Read Sections A.3.1–A.3.3 in the textbook.

Problems

1. Solve Calculational exercises 2, 3 in Chapter 5.

2. In this problem you are asked to solve each of the homogeneous linear systems (a)–(f) below (on the next page, following the worked example) in 3 steps: (i) write the coefficient matrix; (ii) use the Gaussian elimination technique to bring the system to Reduced Row-Echelon Form (RREF) by applying elementary row operations to the coefficient matrix; (iii) using the reduced row-echelon form, write the general form of a solution to the system.

A worked example. Find the general solution of the homogeneous linear system

$$\begin{cases} x_1 + 0x_2 - \frac{1}{2}x_3 - x_4 + x_5 = 0 \\ 2x_1 - x_2 - 3x_3 + 3x_4 - x_5 = 0 \\ 0x_1 + x_2 + 2x_3 + x_4 + 3x_5 = 0 \\ x_1 + 0x_2 - \frac{1}{2}x_3 + x_4 + x_5 = 0 \end{cases}$$

Solution. I start with the matrix of coefficients of the system and perform elementary operations to bring the matrix to RREF form:

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} & -1 & 1 \\ 2 & -1 & -3 & 3 & -1 \\ 0 & 1 & 2 & 1 & 3 \\ 1 & 0 & -\frac{1}{2} & 1 & 1 \end{pmatrix} \xrightarrow[R_4 \leftarrow R_4 - R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 0 & -\frac{1}{2} & -1 & 1 \\ 0 & -1 & -2 & 5 & -3 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow[R_3 \leftarrow R_3 - R_2]{R_2 \leftarrow -R_2} \begin{pmatrix} 1 & 0 & -\frac{1}{2} & -1 & 1 \\ 0 & 1 & 2 & -5 & 3 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \xrightarrow[\substack{R_2 \leftarrow R_2 + 5R_3 \\ R_4 \leftarrow R_4 - 2R_3}]{\substack{R_3 \leftarrow -\frac{1}{6}R_3 \\ R_1 \leftarrow R_1 + R_3}} \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 0 & 1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(Here, $R_2 \leftarrow R_2 - 2R_1$ is the elementary operation of replacing the second row of the matrix by itself minus twice the first row; $R_3 \leftarrow -\frac{1}{6}R_3$ means the elementary operation of multiplying the third row by $-1/6$; etc.) From the RREF, we see that the solution set can be written as

$$\begin{aligned} &\left\{ (\tfrac{1}{2}x_3 - x_5, -2x_3 - 3x_5, x_3, 0, x_5) \ : \ x_3, x_5 \in \mathbb{R} \right\} \\ &= \left\{ x_3(\tfrac{1}{2}, -2, 1, 0, 0) + x_5(1, -3, 0, 0, 1) \ : \ x_3, x_5 \in \mathbb{R} \right\} \\ &= \operatorname{span}\left\{ (\tfrac{1}{2}, -2, 1, 0, 0), (1, -3, 0, 0, 1) \right\} \end{aligned}$$

(a) $\begin{cases} x + 5z = 0 \\ y = 0 \end{cases}$

(b) $\begin{cases} x + 3y + 0z - w = 0 \\ -x - 2y - z + w = 0 \\ y + w = 0 \end{cases}$

(c) $\begin{cases} x + 3y = 0 \\ -x - 2y = 0 \end{cases}$

(d) $\begin{cases} x + 3y = 0 \\ 3x + 9y = 0 \end{cases}$

(e) $\begin{cases} x + y + 5z = 0 \\ y - 10z = 0 \\ 2z = 0 \end{cases}$

(f) $\begin{cases} x + y + 5z = 0 \\ 2x + 3y - 10z = 0 \end{cases}$

3. (a) Solve each of the following *inhomogeneous* linear systems by following the same 3 steps that you used in problem 2 above (except that now the coefficient matrix is an "augmented" matrix). Remember that for inhomogeneous systems, in addition to the possibilities of having a unique solution and infinitely many solutions, there is now also a possibility of having no solutions at all.

(i) $\begin{cases} x + 4z = 2 \\ y = -1 \end{cases}$

(ii) $\begin{cases} 3x - 3y + 15z = -6 \\ x + 2y - z - w = -3 \\ x + 3z + w = -1 \end{cases}$

(iii) $\begin{cases} x + 3y = 1 \\ -x - 2y = 3 \end{cases}$

(iv) $\begin{cases} x + 3y = 0 \\ 3x + 9y = 1 \end{cases}$

(v) $\begin{cases} x + y + 5z = 13 \\ y - 10z = 0 \\ 2z = 4 \end{cases}$

(vi) $\begin{cases} x_1 + 2x_2 + x_3 + 4x_4 + 4x_5 = 5 \\ 2x_1 + 4x_2 - x_3 + 5x_4 + 8x_5 = -5 \\ x_1 + 2x_2 - 2x_3 + x_4 + 4x_5 = -10 \\ x_1 + 2x_2 + 0x_3 + 6x_4 + 8x_5 = 0 \end{cases}$

(b) When solving systems of linear equations, it is always useful to check if the solution you found is correct. Do this for each of the systems above, first of all by going over your computations carefully and looking for errors; second, by testing the solutions you found to see if they actually satisfy the equations. I.e., if you found a unique solution, plug it in to the equations to check they are satisfied. Or, if for example you arrived at a solution of the form

$$\left\{ \begin{pmatrix} -2y+w+3 \\ y \\ 3w+1 \\ w \end{pmatrix} \;\middle|\; y,w\in\mathbb{R} \right\}$$
$$= \left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix} \;\middle|\; y,w\in\mathbb{R} \right\},$$

setting $y = 1, w = 0$ generates the specific solution $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, and setting $y = 0, w = 1$ generates another specific solution $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 4 \\ 1 \end{pmatrix}$. So, plugging these solutions into the system and checking that the equations are satisfied is a good way to test your work.