Homework due. Wednesday 2/26/20 via upload to Canvas.

Reading material. Chapter 8 in the textbook.

Problems

1. Compute the composition $\sigma \circ \pi$ of permutations, where:

   (a) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 2 & 4 & 5 \end{pmatrix}$, $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 1 & 2 & 3 & 4 \end{pmatrix}$

   (b) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$, $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$

2. For each of the permutations σ in the table below (written in the two-line array notation) , calculate its inversion number $\text{inv}(\sigma)$, the sign $\text{sign}(\sigma)$, and the inverse permutation σ^{-1}:

σ	$\text{inv}(\sigma)$	$\text{sign}(\sigma)$	σ^{-1}
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 4 & 2 \end{pmatrix}$			
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 3 & 2 & 4 & 1 & 5 \end{pmatrix}$			
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 1 & 9 & 2 & 3 & 4 & 5 & 7 & 6 & 8 \end{pmatrix}$			

3. Let A be a square matrix of order 4. We perform the following sequence of elementary row operations on A:

   1. Subtract 3 times row 1 from row 2.
   2. Multiply row 3 by 1/5.
   3. Swap rows 1 and 4.
   4. Add 10 times row 1 to row 2.
   5. Swap rows 1 and 2.

   After performing these operations, we get the new matrix $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

   (a) Find a matrix D such that $B = DA$.

   Hint: Each of the elementary row operations can be represented as the operation of multiplying the current matrix from the left by an "elementary matrix".

   (b) Find $\det(A)$.

4. Prove that for any n, the number of permutations of order n with sign 1 is equal to the number of permutations with sign -1 (and hence the two numbers are both equal to $n!/2$, half the total number of permutations of order n).

   Hint: find a way of matching up the permutations of positive and negative signs in pairs, i.e., for each permutation with positive sign find a way to associate with it a permutation with negative sign, such that each "negative" permutation is associated with exactly one "positive" permutation.

5. For each of the following matrices, determine if it is invertible and calculate the inverse matrix if it exists:

$$\text{(a)} \quad \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 0 & -1 & 3 \end{pmatrix} \qquad \text{(b)} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & -1 & 3 \end{pmatrix} \qquad \text{(c)} \quad \begin{pmatrix} 1 & -1 & 6 \\ 2 & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix}$$

   Hint. Use the method based on a variant of Gaussian elimination in which the matrix is augmented by adding the identity matrix to its right, as described in class and discussion section.

6. Solve calculational exercises 1 and 6 in Chapter 8 of the textbook.