Midterm guidelines

The midterm exam will be held on Wednesday February 12 during class time (11:00-11:50). It will be closed book exam — no written material or electronic devices may be used.

Material covered. Chapters 1–5 in the textbook, except for the proof of the Fundamental Theorem of Algebra, and sections A.1-A.3.3 of the textbook appendix.

Midterm format. The exam will have several questions (with each question possibly consisting of several parts). Common types of problems that may appear on the exam include:

- **Word problems.** In a word problem, you are asked to interpret a description in words of a mathematical problem related to a real-life situation and rephrase it in terms of algebraic equations, and then solve those equations to arrive at a conclusion that says something about the real-life situation. Examples of this include the baker problem from the first lecture, homework problem 2 from HW Assignment #1, and problem 1 below.
- **Proof problems.** In a proof problem you will be asked to prove a simple mathematical claim. We saw many examples of this; see also problem 5 below.
- **Calculational problems.** Here, a purely computational challenge is presented (e.g., solving a system of linear equations) and you need to work out the solution. See problems 2, 4, 6 below.
- **Conceptual problems.** A problem asking you about one or more of the concepts we learned in the course (e.g., vector space, linear independence, polynomial, matrix multiplication, etc.) and asking you some questions to test if you understand what the concept is about. This sometimes involves some calculations. See problem 3 below.

Tips for studying. Different people have different approaches to studying, so I can't say what will work best for you, but my suggestions are to combine some amount of each of the following activities:

- Reading the textbook and my lecture summaries.
- Going over your solutions to the homework problems, the TA's critique of your solutions, and the solutions I provided, and making sure you resolve any areas where your understanding is incomplete.
- Practicing solving additional problems in the textbook from the ones I did not assign as homework, and other problems you find in other books or online.
- Solving the practice problems provided below.

Happy studying!

Practice problems

Note. The problems below are provided for your use as practice problems to test your skills and understanding of the material against, but they are not a "practice exam", in the sense that they are not designed to be similar to the actual midterm in terms of their difficulty level, the amount of time you should need to solve them, or the specific topics they cover. So please don't make any assumptions based on these problems that I will be asking about subject X or that the questions you will be asked must be of a certain specific form.

1. In chemistry, one encounters the chemical reaction

$$\mathrm{NaOH} + \mathrm{H_2SO_4} \quad \longrightarrow \quad \mathrm{Na_2SO_4} + \mathrm{H_2O}$$

   in which sodium hydroxide (NaOH) reacts with sulfuric acid (H_2SO_4) to yield sodium sulfate (Na_2SO_4) and water (H_2O).

   From the chemical symbols we see that sodium hydroxide contains one atom of sodium (Na), one atom of oxygen (O), and one atom of hydrogen (H); sulfuric acid contains two atoms of hydrogen, one of sulphur (S) and four of oxygen; sodium sulfate contains two atoms of sodium, one of sulphur and four of oxygen, and water contains two atoms of hydrogen and one of oxygen.

   The chemical equation above is not "balanced," since the number of atoms of each of the four types participating in the reaction differ on the left and right hand sides (for example, there is only one "Na" on the left but two on the right). Use linear algebra to balance the equation; that is, find integers x_1, x_2, x_3 and x_4 for which the more precise equation

$$x_1\mathrm{NaOH} + x_2\mathrm{H_2SO_4} \quad \longrightarrow \quad x_3\mathrm{Na_2SO_4} + x_4\mathrm{H_2O}$$

   is satisfied; this refers to a chemical reaction in which x_1 molecules of sodium hydroxide react with x_2 molecules of sulfuric acid to produce x_3 molecules of sodium sulfate and x_4 molecules of water.

   Guidance: the idea is that the number of atoms of each type must be equal before and after the reaction. This leads to linear equations for x_1, x_2, x_3 and x_4, which can be solved. (Look for a solution where x_1, x_2, x_3, x_4 are very small integers.)

2. Perform the following calculations involving complex numbers:

   (a) $\dfrac{3-2i}{1+i} =$

   (b) $\sqrt{-i} =$

   (c) $(\sqrt{3}+i)^6 =$

   (d) If $p(z)$ is the polynomial $p(z) = z^4 + z^3 - 7z^2 - z + 6$, find coefficients a, b, c such that $p(z)$ has the factorization $p(z) = (z^2-1)(az^2+bz+c)$. Use this to find all complex solutions to the equation $p(z) = 0$.

3. Let P_2 denote the vector space of polynomials that are of the form $p(x) = ax^2 + bx + c$ with real coefficients a, b, c. Define elements $p_1(x), p_2(x), p_3(x), p_4(x)$ of P_2 by

$$\begin{aligned} p_1(x) &= x^2 + 2, \\ p_2(x) &= 2x + 3, \\ p_3(x) &= \alpha x^2 + 6, \\ p_4(x) &= x^2 + 2x + 5, \end{aligned}$$

where α is a parameter.

(a) For which values of α are the polynomials p_1, p_2, p_3 linearly independent, considered as elements of the vector space P_2? Explain how you arrived at the answer.

(b) For which values of α are the polynomials p_1, p_2, p_3, p_4 linearly independent, considered as elements of P_2? Explain.

(c) For which values of α do the polynomials p_1, p_2, p_3, p_4 span the space P_2? Explain.

4. Use the Gaussian elimination method to find the general solution to the homogeneous system of linear equations

$$\begin{cases} 3x + 8y + 1z - 12w - 4u - 2v = 0 \\ x - w = 0 \\ 4x + 4y + 3z - 40w - 3u - v = 0 \\ 2y + z - 3w - 2u = 0 \\ y + - 12w = 0 \end{cases}$$

Note: in case you are curious, this system arises in a similar way to the system in problem 1 above out of the problem of balancing the chemical reaction

$$H_3PO_4 + (NH_4)_2MoO_4 + HNO_3 \longrightarrow (NH_4)_3PO_4 + 12MoO_3 + NH_4NO_3 + H_2O.$$

(See the web page `https://www.wikihow.com/Balance-Chemical-Equations-Using-Linear-Algebra`.)

5. Solve proof-writing exercises 5, 7, and 8 in Chapter 5 of the textbook.

6. Define three matrices A, B, C by

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In each entry of the table below, fill in the matrix multiplication of the two matrices associated with the table row and column (with the table represented by the row on the left and the table represented by the column on the right, so that for example the entry in the rightmost column of the first row would correspond to the matrix multiplication AC), or write "undefined" if the multiplication of those two matrices (in that order) is not defined.

	A	B	C
A			
B			
C			