Using the Mathematica package *OrientedSwaps*

This notebook illustrates the use of the *OrientedSwaps* package.

Loading the package and getting help

10/17/19 22:08:12 In[1]:=

```
SetDirectory["Desktop"]; (* Set the working directory to
 the folder where we placed the package file OrientedSwaps.m *)
```

10/17/19 22:08:17 In[3]:=

```
<< orientedswaps.m  (* Load the package *)
```

```
OrientedSwaps.m package (Version 1.0) loaded. Type: 'OrientedSwapsHelp[]' for help.
```

10/17/19 22:08:18 In[4]:=

```
OrientedSwapsHelp[]  (* This prints out a help message *)
```

OrientedSwaps (Version 1.0, October 17, 2019)

OrientedSwaps is a companion Mathematica package to the 2019 paper
"Sorting networks, staircase shape Young tableaux and last passage percolation",
by Elia Bisi, Fabio Deelan Cunden, Shane Gibbons and Dan Romik

The goal of this package is to verify
the conjectural identity between the two generating functions
$F_n(x_1, . . . , x_{n-1})$ and $G_n(x_1, . . . , x_{n-1})$ defined in the paper, for small values of n
.

To verify the identity, run the following command:

```
genfuns = VerifyStaircaseSYTSortingNetworksIdentity[n];
```

where n is 3, 4, 5 or 6. The calculations for n=3,4,5 are almost instantaneous.
The calculation for n=6 will take around 10-20 minutes on a modern machine.

The output variable genfuns will contain a list of the pairs of components
of the two generating functions, indexed by permutations of order n-1.
Feel free to inspect it and try to understand why the identity is true.

Verifying the identity in the case $n = 3$

10/17/19 22:08:20 In[5]:=

```
genfuns = VerifyStaircaseSYTSortingNetworksIdentity[3];
```

```
The conjecture is TRUE for n=3 (summed
  over 2 staircase shape tableaux and sorting networks)
```

10/17/19 22:08:23 In[6]:=

```
genfuns (* Inspect the two generating functions *)
```

10/17/19 22:08:23 Out[6]=

$$\left\{\left\{\{1,2\}, \frac{1}{(1+x_1)\,(2+x_1)\,(1+x_2)}, \frac{1}{(1+x_1)\,(2+x_1)\,(1+x_2)}\right\}, \left\{\{2,1\}, \frac{1}{(1+x_1)\,(2+x_1)\,(1+x_2)}, \frac{1}{(1+x_1)\,(2+x_1)\,(1+x_2)}\right\}\right\}$$

Verifying the identity in the case $n = 4$

10/17/19 22:08:26 In[7]:=

```
genfuns = VerifyStaircaseSYTSortingNetworksIdentity[4];
```

```
The conjecture is TRUE for n=4 (summmed
  over 16 staircase shape tableaux and sorting networks)
```

10/17/19 22:08:26 In[8]:=

```
genfuns // Column   (* Inspect the two generating functions *)
```

10/17/19 22:08:26 Out[8]=

$$\left\{\{1,2,3\}, \frac{2}{(1+x_1)\,(2+x_1)^2\,(3+x_1)\,(2+x_2)\,(1+x_3)} + \frac{1}{(1+x_1)\,(2+x_1)^2\,(1+x_2)\,(2+x_2)\,(1+x_3)}, \frac{2}{(1+x_1)\,(2+x_1)^2\,(3+x_1)\,(1+x_2)\,(1+x_3)} + \frac{1}{(2+x_1)^2\,(3+x_1)\,(1+x_2)\,(2+x_2)\,(1+x_3)}\right\}$$

$$\left\{\{1,3,2\}, \frac{2}{(1+x_1)\,(2+x_1)^2\,(3+x_1)\,(2+x_2)\,(1+x_3)} + \frac{1}{(1+x_1)\,(2+x_1)^2\,(1+x_2)\,(2+x_2)\,(1+x_3)}, \frac{2}{(1+x_1)\,(2+x_1)^2\,(3+x_1)\,(1+x_2)\,(1+x_3)} + \frac{1}{(2+x_1)^2\,(3+x_1)\,(1+x_2)\,(2+x_2)\,(1+x_3)}\right\}$$

$$\left\{\{2,1,3\}, \frac{2}{(1+x_1)\,(2+x_1)^2\,(3+x_1)\,(2+x_2)\,(1+x_3)}, \frac{2}{(1+x_1)\,(2+x_1)^2\,(3+x_1)\,(2+x_2)\,(1+x_3)}\right\}$$

$$\left\{\{2,3,1\}, \frac{2}{(1+x_1)\,(2+x_1)^2\,(3+x_1)\,(2+x_2)\,(1+x_3)} + \frac{1}{(1+x_1)\,(2+x_1)^2\,(1+x_2)\,(2+x_2)\,(1+x_3)}, \frac{2}{(1+x_1)\,(2+x_1)^2\,(3+x_1)\,(1+x_2)\,(1+x_3)} + \frac{1}{(2+x_1)^2\,(3+x_1)\,(1+x_2)\,(2+x_2)\,(1+x_3)}\right\}$$

$$\left\{\{3,1,2\}, \frac{2}{(1+x_1)\,(2+x_1)^2\,(3+x_1)\,(2+x_2)\,(1+x_3)}, \frac{2}{(1+x_1)\,(2+x_1)^2\,(3+x_1)\,(2+x_2)\,(1+x_3)}\right\}$$

$$\left\{\{3,2,1\}, \frac{2}{(1+x_1)\,(2+x_1)^2\,(3+x_1)\,(2+x_2)\,(1+x_3)} + \frac{1}{(1+x_1)\,(2+x_1)^2\,(1+x_2)\,(2+x_2)\,(1+x_3)}, \frac{2}{(1+x_1)\,(2+x_1)^2\,(3+x_1)\,(1+x_2)\,(1+x_3)} + \frac{1}{(2+x_1)^2\,(3+x_1)\,(1+x_2)\,(2+x_2)\,(1+x_3)}\right\}$$

Verifying the identity in the case $n = 5$

10/17/19 22:08:28 In[9]:=

```
genfuns = VerifyStaircaseSYTSortingNetworksIdentity[5];
```

```
The conjecture is TRUE for n=5 (summmed
  over 768 staircase shape tableaux and sorting networks)
```

A complete printout of the two generating functions in this case will take up a lot of space (they each have 24 components corresponding to the 24 permutations of order 4). Here is an example of just one of the components of the two generating functions.

10/17/19 22:08:32 In[10]:=

```
{genfuns[[1]][[1]], "  ", genfuns[[1]][[2]], "  ", genfuns[[1]][[3]]} // Column
```

10/17/19 22:08:32 Out[10]=

{1, 2, 3, 4}

$$\frac{8}{(1+x_1)(2+x_1)^2(3+x_1)^3(4+x_1)(3+x_2)(2+x_3)(1+x_4)}+\frac{8}{(1+x_1)(2+x_1)^3(3+x_1)^2(4+x_1)(3+x_2)(2+x_3)(1+x_4)}+$$
$$\frac{4}{(1+x_1)(2+x_1)^2(3+x_1)^2(2+x_2)^2(3+x_2)(2+x_3)(1+x_4)}+\frac{2}{(1+x_1)(2+x_1)^3(3+x_1)(2+x_2)^2(3+x_2)(2+x_3)(1+x_4)}+$$
$$\frac{2}{(1+x_1)(2+x_1)^3(1+x_2)(2+x_2)^2(3+x_2)(2+x_3)(1+x_4)}+\frac{6}{(1+x_1)(2+x_1)^2(3+x_1)^3(2+x_2)(3+x_2)(2+x_3)(1+x_4)}+$$
$$\frac{5}{(1+x_1)(2+x_1)^3(3+x_1)^2(2+x_2)(3+x_2)(2+x_3)(1+x_4)}+\frac{2}{(1+x_1)(2+x_1)^2(3+x_1)^2(2+x_2)^2(1+x_3)(2+x_3)(1+x_4)}+$$
$$\frac{1}{(1+x_1)(2+x_1)^3(3+x_1)(2+x_2)^2(1+x_3)(2+x_3)(1+x_4)}+\frac{1}{(1+x_1)(2+x_1)^3(1+x_2)(2+x_2)^2(1+x_3)(2+x_3)(1+x_4)}+$$
$$\frac{2}{(1+x_1)(2+x_1)^2(3+x_1)^3(2+x_2)(1+x_3)(2+x_3)(1+x_4)}+\frac{3}{(1+x_1)(2+x_1)^3(3+x_1)^2(2+x_2)(1+x_3)(2+x_3)(1+x_4)}$$

$$\frac{8}{(1+x_1)(2+x_1)^2(3+x_1)^3(4+x_1)(1+x_2)(1+x_3)(1+x_4)}+\frac{8}{(1+x_1)(2+x_1)^3(3+x_1)^2(4+x_1)(1+x_2)(1+x_3)(1+x_4)}+$$
$$\frac{2}{(2+x_1)(3+x_1)^3(4+x_1)(1+x_2)(2+x_2)^2(1+x_3)(1+x_4)}+\frac{4}{(2+x_1)^2(3+x_1)^2(4+x_1)(1+x_2)(2+x_2)^2(1+x_3)(1+x_4)}+$$
$$\frac{5}{(2+x_1)^2(3+x_1)^3(4+x_1)(1+x_2)(2+x_2)(1+x_3)(1+x_4)}+\frac{6}{(2+x_1)^3(3+x_1)^2(4+x_1)(1+x_2)(2+x_2)(1+x_3)(1+x_4)}+$$
$$\frac{2}{(3+x_1)^3(4+x_1)(1+x_2)(2+x_2)^2(3+x_2)(1+x_3)(1+x_4)}+\frac{1}{(2+x_1)(3+x_1)^3(4+x_1)(2+x_2)^2(1+x_3)(2+x_3)(1+x_4)}+$$
$$\frac{2}{(2+x_1)^2(3+x_1)^2(4+x_1)(2+x_2)^2(1+x_3)(2+x_3)(1+x_4)}+\frac{3}{(2+x_1)^2(3+x_1)^3(4+x_1)(2+x_2)(1+x_3)(2+x_3)(1+x_4)}+$$
$$\frac{2}{(2+x_1)^3(3+x_1)^2(4+x_1)(2+x_2)(1+x_3)(2+x_3)(1+x_4)}+\frac{1}{(3+x_1)^3(4+x_1)(2+x_2)^2(3+x_2)(1+x_3)(2+x_3)(1+x_4)}$$