Math 25 — Practice problems for the final

Important notes

- The final will be **2 hours long**.
- The final will be a **closed-book exam**.
- **All of the course material will be covered** (except parts that were explicitly described as enrichment material, e.g., the proof that e is irrational), with an emphasis on material covered after the midterm.
- The practice problems in this problem set are designed to aid you in studying and reviewing important parts of the course material. They are designed to be in some cases slightly more difficult, and to take longer to solve, than actual exam questions.
- The practice problems do not cover all the course topics. Topics that are not covered by this problem set may still appear on the final!
- Solutions to this problem set will be posted on the course web page later this week. (An email announcement will be sent.)
- It is also recommended to go over the homework and its solutions (as well as the textbook and class notes) as preparation for the final.

One final recommendation for the exam

- When writing a proof, **use words** to explain the logic of what you are doing — don't just write formulas or equations (like you may be used to doing in other math classes). In other words, remember the following rules:

$$\text{Proof} \neq \text{formulas}$$

$$\text{Proof} = \text{formulas} + \text{explanation}$$

Question 1

For $n = 0, 1, 2, 3, \ldots$ and $0 \leq k \leq n$ define numbers C_k^n by

$$C_k^n = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\ldots(n-k+1)}{k!}$$

(for $k = 0$ and $k = n$ we define $C_0^n = C_n^n = 1$).

(a) Prove that for all $n \geq 1$ and $1 \leq k \leq n-1$,

$$C_k^n = C_{k-1}^{n-1} + C_k^{n-1}$$

Hint: $\frac{n}{k} = 1 + \frac{n-k}{k}$

(b) Prove by induction on n that for all $n \geq 1$,

$$\sum_{k=0}^{n} C_k^n = C_0^n + C_1^n + C_2^n + \ldots + C_n^n = 2^n$$

Question 2

(a) Let $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$ be sequences of real numbers. For each of the following identities, explain what assumptions are needed to ensure that the identity is valid:

i. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

ii. $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

iii. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

(b) Find the limit L of the sequence given by

$$a_n = \frac{5n^4 + 3n^2 - 10}{(2n^2 + \sin(n))^2}$$

Prove rigorously that $L = \lim_{n\to\infty} a_n$, by appealing either to the definition of the limit or to known results about limits.

Question 3

(a) State the squeeze theorem.

(b) Denote $a_n = \sqrt[n]{n}$. Prove that $\lim_{n\to\infty} a_n = 1$. You may use the fact that the inequality $(1+x)^n \geq \frac{n(n-1)}{2}x^2$ holds for all $n \geq 1$ and $x > 0$.

Question 4

(a) Prove that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(b) Let p be a positive real number. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

(c) Prove that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log_2 n}$ diverges (here, $\log_2 n = \ln(n)/\ln(2)$ denotes the base-2 logarithm of n).

Question 5

(a) Evaluate the infinite series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots$$

(b) Evaluate the infinite series

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$$

(c) Evaluate the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \dots$$

Question 6

Give an example of:

(a) A divergent series.

(b) An absolutely convergent series.

(c) A conditionally convergent series.

(d) A series of the form $\sum_{n=1}^{\infty}(a_n + b_n)$ that is convergent but such that both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent.

(e) A series of the form $\sum_{n=1}^{\infty}(a_n \cdot b_n)$ that is convergent but such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent.

(f) A divergent series with bounded partial sums.

* (g) A series $\sum_{n=1}^{\infty} a_n$ that is divergent such that $\lim_{n\to\infty}(n \cdot a_n) = 0$ (i.e., loosely speaking, the sequence being summed converges to 0 faster than $1/n$).

* = more difficult

Question 7

(a) Prove that $\lim_{n\to\infty} \frac{n}{2^n} = 0$. You may use the fact that the inequality $(1+x)^n \geq \frac{n(n-1)}{2}x^2$ holds for all $n \geq 1$ and $x > 0$.

(b) Prove that $\sum_{n=1}^{\infty} \frac{1}{2^n - n}$ converges.

Question 8

The goal of this problem is to compute the value of the infinite sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \dots$$

(and in particular to show that it converges, which strengthens the result of question 7(a) above).

(a) Define a new sequence $(x_n)_{n=1}^{\infty}$ whose terms are given by

$$(x_n)_{n=1}^{\infty} = \left\{\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{32}, \tfrac{1}{32}, \tfrac{1}{32}, \tfrac{1}{32}, \tfrac{1}{32}, \dots\right\}$$

Explain why if $\sum_{n=1}^{\infty} x_n$ converges, then $\sum_{n=1}^{\infty} \frac{n}{2^n}$ also converges and the values of the two series are the same.

Hint: The sequence of partial sums of $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is a subsequence of the sequence of partial sums of $\sum_{n=1}^{\infty} x_n$.

(b) Rearrange the terms of $\sum_{n=1}^{\infty} x_n$ by writing the sum as

$$\left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} \dots\right) + \left(\tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \dots\right) + \left(\tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \dots\right) + \dots$$

(This part is more of a "thinking question", requiring you to think about why this rearrangement makes sense and why it is valid to perform it).

(c) Evaluate each of the internal sums in the above rearrangement, and the sum of their values, to conclude that $S = 2$.