# An approximation formula in Hilbert space *

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**Abstract** Let $\{\varphi_k\}_1^\infty$ be a frame for Hilbert space $H$. The purpose of this paper is to present an approximation formula of any $f \in H$ by a linear combination of finitely many frame elements in the frame $\{\varphi_k\}_1^\infty$ and show that the obtained approximation error depends on the bounds of frame and the convergence rate of frame coefficients of $f$ as well as the relation among frame elements.

**Key words:** approximation formula; linear combination; frame element; Hilbert space

**MSC** 42C15

## 1. Introduction

Let $H$ be a Hilbert space and $\{e_k\}_1^\infty$ be an orthonormal basis for $H$. It is well-known that any $f \in H$ can be approximated by the linear combination of $\{e_k\}_1^n$ and the approximation error depends on the convergence rate of the Fourier coefficients[2].

As a generalization of the orthonormal bases, Duffin and Schaeffer[3] introduced the notion of frames. Suppose that $\{\varphi_k\}_1^\infty$ is a frame for $H$. For any $f \in H$, we will construct a linear combination of finitely many frame elements in the frame $\{\varphi_k\}_1^\infty$ to approximate to $f$ and show that the approximation error depends on the bounds of frame and the convergence rate of frame coefficients of $f$ as well as the relation among frame elements.

We recall some concepts and propositions.

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Let $\{\varphi_k\}_1^\infty$ be a sequence in Hilbert space $H$. If there exist two positive constants $A,\ B$ such that

$$A \parallel f \parallel^2 \leq \sum_{k=1}^{\infty} |(f,\varphi_k)|^2 \leq B \parallel f \parallel^2 \quad \forall f \in H, \tag{1.1}$$

then the sequence $\{\varphi_k\}_1^\infty$ is said to be a frame for $H$, where $A$ and $B$ are said to be frame bounds. Specially, if $A = B = 1$ and $\parallel \varphi_k \parallel = 1$ $(k \in Z^+)$, then $\{\varphi_k\}_1^\infty$ is an orthonormal basis for $H$.

Let $\{\varphi_k\}_1^\infty$ be a frame for $H$. The frame operator $S$ is defined as

$$S: \quad H \to H, \qquad Sf = \sum_{k=1}^{\infty} (f,\varphi_k)\varphi_k \quad \forall f \in H, \tag{1.2}$$

where $(f,\varphi_k)$ $(k \in Z^+)$ are said to be frame coefficients.

**Proposition 1.1**[1,p58-62]. Let $\{\varphi_k\}_1^\infty$ be a frame with bounds $A$ and $B$ for $H$. Then

(i) The frame operator $S$ is a self-conjugate operator and $\quad A \parallel f \parallel \leq \parallel Sf \parallel \leq B \parallel f \parallel \quad \forall f \in H.$

(ii) The inverse operator $S^{-1}$ exists and $\quad \frac{1}{B} \parallel f \parallel \leq \parallel S^{-1} f \parallel \leq \frac{1}{A} \parallel f \parallel \quad \forall f \in H.$

(iii) Denote $\widetilde{\varphi}_k = S^{-1}\varphi_k$, the $\{\widetilde{\varphi}_k\}_1^\infty$ is also a frame for $H$ and

$$\frac{1}{B} \parallel f \parallel^2 \leq \sum_{k=1}^{\infty} |(f,\widetilde{\varphi}_k)|^2 \leq \frac{1}{A} \parallel f \parallel^2 \quad \forall f \in H. \tag{1.3}$$

(iv) For each $k$,

$$\widetilde{\varphi}_k = \frac{2}{A+B}\varphi_k + \frac{2}{A+B}\sum_{n=1}^{\infty}\left(I - \frac{2S}{A+B}\right)^n \varphi_k, \tag{1.4}$$

where $I$ is the identity operator.

(v) Let $R = I - \frac{2S}{A+B}$. Then $\parallel R \parallel \leq \frac{B-A}{B+A}$.

The frame $\{\widetilde{\varphi}_k\}_1^\infty$ is said to be a dual frame of $\{\varphi_k\}_1^\infty$.

Denote the partial sum of the series in (1.4) by $\widetilde{\varphi}_k^N$, i.e.

$$\widetilde{\varphi}_k^0 = \frac{2}{A+B}\varphi_k, \qquad \widetilde{\varphi}_k^N = \frac{2}{A+B}\varphi_k + \frac{2}{A+B}\sum_{n=1}^{N}\left(I - \frac{2S}{A+B}\right)^n \varphi_k. \tag{1.5}$$

**Proposition 1.2**[1,p58-62]. Under the conditions of Proposition 1.1, then

(i) for any $f \in H$, the reconstruction formula $\ f = \sum\limits_{k=1}^{\infty} (f,\varphi_k)\widetilde{\varphi}_k\ $ holds

(ii) for any $f \in H$, the series $\sum\limits_{k=1}^{\infty}(f,\varphi_k)\widetilde{\varphi}_k^N$ is convergent and

$$\| f - \sum_{k=1}^{\infty}(f,\varphi_k)\widetilde{\varphi}_k^N \| \leq q^{N+1} \| f \| \quad (N \in Z^+), \tag{1.6}$$

where $q = \frac{B-A}{B+A}$.

## 2. A new frame approximation operator $(\sigma_n^N(\cdot))_m$

In order to approximate to any $f \in H$ by a linear combination of finitely many frame elements, we present a new frame approximation operator in this section.

Let $\{\varphi_k\}_1^\infty$ be a frame for $H$ with bounds $A$ and $B$. The frame operator $S$, $\widetilde{\varphi}_k$, and $\widetilde{\varphi}_k^N$ are stated in (1.2), (1.4), and (1.5), respectively.

**Definition 2.1** We truncate the series $\sum\limits_{k=1}^{\infty}(f,\varphi_k)\widetilde{\varphi}_k^N$ by its partial sum, for $n,\ N \in Z^+$, define

$$\sigma_n^N(f) := \sum_{k=1}^{n}(f,\varphi_k)\widetilde{\varphi}_k^N \quad \forall f \in H. \tag{2.1}$$

Denote

$$S_m: \quad H \to H, \qquad S_m f := \sum_{j=1}^{m}(f,\varphi_j)\varphi_j \quad \forall f \in H. \tag{2.2}$$

and

$$\begin{aligned} &S_m^1 := S_m, \qquad S_m^l := S_m^{l-1}(S_m) \quad (l \in Z^+),\\ &S^1 := S, \qquad S^l := S^{l-1}(S) \qquad (l \in Z^+). \end{aligned} \tag{2.3}$$

**Definition 2.2.** For any $k,\ N,\ m \in Z^+$, define

$$(\widetilde{\varphi}_k^0)_m = \frac{2}{A+B}\varphi_k \quad \text{and} \quad (\widetilde{\varphi}_k^N)_m = \frac{2}{A+B}\varphi_k + \frac{2}{A+B}\sum_{n=1}^{N}\left(I - \frac{2S_m}{A+B}\right)^n \varphi_k.$$

From this definition, we have

$$(\widetilde{\varphi}_k^N)_m = (\widetilde{\varphi}_k^{N-1})_m + \frac{2}{A+B}\left(I - \frac{2S_m}{A+B}\right)^N \varphi_k \quad (N \in Z^+). \tag{2.4}$$

**Definition 2.3.** Define a frame approximation operator as follows.

$$(\sigma_n^N(\cdot))_m: \quad H \to H, \qquad (\sigma_n^N(f))_m = \sum_{k=1}^{n}(f,\varphi_k)(\widetilde{\varphi}_k^N)_m \quad \forall f \in H.$$

**Lemma 2.4.** (i) The sequence of operators $\{S_m\}_1^\infty$ is uniformly bounded.

(ii) For any $f \in H$,

$$S_m^l f \to S^l f \ (m \to \infty) \quad \forall l \in Z^+. \tag{2.5}$$

**Proof.** Combining (1.2) with (2.2), for any $f \in H$, we have

$$S_m f \to Sf \quad (m \to \infty). \tag{2.6}$$

Using the resonance theorem, we know that there exists $M > 0$ such that $\| S_m \| \leq M$ $(m \in Z^+)$. So we get (i).

We have known that (2.5) holds for $l = 1$. Now we assume that (2.5) holds for $l - 1$.

Noticing that $S_m^l f - S^l f = S_m(S_m^{l-1} f) - S_m(S^{l-1} f) + S_m(S^{l-1} f) - S(S^{l-1} f)$, we have

$$\| S_m^l f - S^l f \| \leq \| S_m(S_m^{l-1} f - S^{l-1} f) \| + \| S_m(S^{l-1} f) - S(S^{l-1} f) \| =: p_m^{(l)}(f) + q_m^{(l)}(f). \tag{2.7}$$

Since $\| S_m \| \leq M$, by the postulate of induction, we have

$$p_m^{(l)}(f) \leq M \| S_m^{l-1} f - S^{l-1} f \| \to 0 \quad (m \to \infty).$$

Let $g = S^{l-1} f$. Then by (2.6),

$$q_m^{(l)}(f) = \| S_m g - Sg \| \to 0 \ (m \to \infty).$$

Hence, by (2.7), we have $S_m^l f \to S^l f$ $(m \to \infty)$, i.e. (2.5) holds for any $l \in Z^+$. So we get (ii).

**Lemma 2.5.** For any $f \in H$, $(\sigma_n^N(f))_m$ is a linear combination of $\{\varphi_j\}_1^\lambda$, where $\lambda = \max\{m, n\}$.

**Proof.** By Definition 2.2 and the operator equality

$$(I - \frac{2S_m}{A+B})^n = I + \sum_{l=1}^{n} \begin{pmatrix} n \\ l \end{pmatrix} \left(-\frac{2}{A+B}\right)^l S_m^l,$$

we conclude that

$$(\widetilde{\varphi}_k^N)_m = \frac{2(N+1)}{A+B}\varphi_k + \frac{2}{A+B}\sum_{n=1}^{N}\sum_{l=1}^{n} \begin{pmatrix} n \\ l \end{pmatrix} \left(-\frac{2}{A+B}\right)^l S_m^l \varphi_k.$$

Again by Definition 2.3, we get

$$(\sigma_n^N(f))_m = \frac{2(N+1)}{A+B}\sum_{k=1}^{n}(f, \varphi_k)\varphi_k + \frac{2}{A+B}\sum_{n=1}^{N}\sum_{l,k=1}^{n} b_{n,l,k} S_m^l \varphi_k =: M_1 + M_2,$$

where $b_{n,l,k} = \begin{pmatrix} n \\ l \end{pmatrix} \left(-\frac{2}{A+B}\right)^l (f, \varphi_k)$.

For any $f \in H$, by (2.2), we obtain that for any $k,\ l \in Z^+$,

$$S_m^l \varphi_k = \sum_{j=1}^{m} c_j \varphi_j, \qquad \text{where} \qquad c_j = \sum_{\nu_1,\ldots,\nu_{l-1}=1}^{m} (\varphi_k, \varphi_{\nu_1})(\varphi_{\nu_1}, \varphi_{\nu_2}) \cdots (\varphi_{\nu_{l-1}}, \varphi_j).$$

So we see that for $l,\ k \in Z^+$, $S_m^l \varphi_k$ is a linear combination of $m$ frame elements $\varphi_1, \varphi_2, ..., \varphi_m$, further, the sum $M_2$ is a linear combination of $m$ elements $\varphi_1, \varphi_2, ..., \varphi_m$. Clearly, the sum $M_1$ is a linear combination of $n$ elements $\varphi_1, \varphi_2, ..., \varphi_n$. Therefore, $(\sigma_n^N(f))_m$ is a linear combination of $\{\varphi_j\}_1^\lambda$, where $\lambda = \max\{m,\ n\}$. Lemma 2.5 is proved.

## 3. Approximation by $(\sigma_n^N(\cdot))_m$

We will approximate to $f$ by $(\sigma_n^N(f))_m$. First, we estimate $\| f - \sigma_n^N(f) \|$ in Lemma 3.1. Next, we estimate $\| \sigma_n^N(f) - (\sigma_n^N(f))_m \|$ in Lemma 3.3. Finally, we get an estimate $\| f - (\sigma_n^N(f))_m \|$ in Theorem 3.4. Meanwhile, we show that the approximation error only depends on the frame bounds and the convergence rate of the frame coefficients of $f$ as well as the relation among frame elements.

**Lemma 3.1.** Let $\{\varphi_k\}_1^\infty$ be a frame for $H$ with bounds $A,\ B$ and $\sigma_n^N(f)$ be stated in (2.1). Denote

$$\varepsilon_n(f) := \frac{1}{\| f \|} \left( \sum_{j=n+1}^{\infty} |(f, \varphi_j)|^2 \right)^{\frac{1}{2}}. \tag{3.1}$$

Then for any $f \in H$, we have

$$\| f - \sigma_n^N(f) \| \leq q^{N+1} \| f \| + \frac{1}{\sqrt{A}} \left(1 + q^{N+1}\right) \| f \| \varepsilon_n(f) \quad (n,\ N \in Z^+), \tag{3.2}$$

where $q = \frac{B-A}{B+A}$.

**Remark 3.2.** Since $\{\varphi_k\}_1^\infty$ is a frame, we see that $\sum_{k=1}^{\infty} |(f, \varphi_k)|^2 < \infty$. From this and (3.1), we get $\varepsilon_n(f) \to 0$ $(n \to \infty)$.

**Proof of Lemma 3.1.** By Proposition 1.2(ii) and (2.1), we know that for any $f \in H$ and $N \in Z^+$, the series $\sum_{k=1}^{\infty} (f, \varphi_k) \widetilde{\varphi}_k^N$ converges and $\sigma_n^N(f) = \sum_{k=1}^{n} (f, \varphi_k) \widetilde{\varphi}_k^N$ is its partial sum. Denote its remainder term by

$$r_n^N(f) := \sum_{k=n+1}^{\infty} (f, \varphi_k) \widetilde{\varphi}_k^N.$$

So

$$\| f - \sigma_n^N(f) \| = \| f - \left( \sum_{k=1}^{\infty} (f, \varphi_k)\widetilde{\varphi}_k^N - r_n^N(f) \right) \| \leq \| f - \sum_{k=1}^{\infty} (f, \varphi_k)\widetilde{\varphi}_k^N \| + \| r_n^N(f) \| .$$

Using Proposition 1.2(ii), we get

$$\| f - \sigma_n^N(f) \| \leq q^{N+1} \| f \| + \| r_n^N(f) \| . \tag{3.3}$$

By Proposition 1.2(i), the series $\sum_{k=1}^{\infty} (f, \varphi_k)\widetilde{\varphi}_k$ converges, so we can decompose $r_n^N(f)$ as follows:

$$r_n^N(f) = \sum_{k=n+1}^{\infty} (f, \varphi_k)\widetilde{\varphi}_k + \sum_{k=n+1}^{\infty} (f, \varphi_k)(\widetilde{\varphi}_k^N - \widetilde{\varphi}_k) =: u_n^N(f) + v_n^N(f). \tag{3.4}$$

By (1.4) and (1.5), it follows that

$$\begin{aligned}
\widetilde{\varphi}_k - \widetilde{\varphi}_k^N &= \tfrac{2}{A+B} \sum_{n=N+1}^{\infty} (I - \tfrac{2S}{A+B})^n \varphi_k \\
&= (I - \tfrac{2S}{A+B})^{N+1} \left( \tfrac{2}{A+B}\, \varphi_k + \tfrac{2}{A+B} \sum_{n=1}^{\infty} (I - \tfrac{2S}{A+B})^n \varphi_k \right) \\
&= (I - \tfrac{2S}{A+B})^{N+1} \widetilde{\varphi}_k = R^{N+1} \widetilde{\varphi}_k,
\end{aligned}$$

where $R = I - \frac{2S}{A+B}$. So we get

$$v_n^N(f) = - \sum_{k=n+1}^{\infty} (f, \varphi_k) R^{N+1} \widetilde{\varphi}_k = -R^{N+1} \left( \sum_{k=n+1}^{\infty} (f, \varphi_k)\widetilde{\varphi}_k \right).$$

From this and (3.4), we get

$$r_n^N(f) = \sum_{k=n+1}^{\infty} (f, \varphi_k)\widetilde{\varphi}_k - R^{N+1} \left( \sum_{k=n+1}^{\infty} (f, \varphi_k)\widetilde{\varphi}_k \right) = (I - R^{N+1}) \sum_{k=n+1}^{\infty} (f, \varphi_k)\widetilde{\varphi}_k. \tag{3.5}$$

However, we have

$$\| \sum_{k=n+1}^{\infty} (f, \varphi_k)\widetilde{\varphi}_k \|^2 = \sup_{\|g\|=1} \left| \left( \sum_{k=n+1}^{\infty} (f, \varphi_k)\widetilde{\varphi}_k, \;\; g \right) \right|^2 = \sup_{\|g\|=1} \left| \sum_{k=n+1}^{\infty} (f, \varphi_k)(\widetilde{\varphi}_k, g) \right|^2 .$$

Using Cauchy's inequality in $l^2$, we get

$$\| \sum_{k=n+1}^{\infty} (f, \varphi_k)\widetilde{\varphi}_k \|^2 \leq \left( \sum_{k=n+1}^{\infty} |(f, \varphi_k)|^2 \right) \cdot \sup_{\|g\|=1} \left( \sum_{k=n+1}^{\infty} |(\widetilde{\varphi}_k, g)|^2 \right).$$

By Proposition 1.1 (iii), we know that $\{\widetilde{\varphi}_k\}_1^\infty$ is also a frame for $H$ and

$$\sum_{k=1}^{\infty} |(\widetilde{\varphi}_k, g)|^2 \leq \frac{1}{A} \parallel g \parallel^2 .$$

From this, we get

$$\parallel \sum_{k=n+1}^{\infty} (f, \varphi_k)\widetilde{\varphi}_k \parallel^2 \leq \frac{1}{A} \left( \sum_{k=n+1}^{\infty} |(f, \varphi_k)|^2 \right).$$

Again by (3.1) and (3.5), we have

$$\parallel r_n^N(f) \parallel \leq \parallel I - R^{N+1} \parallel \cdot \left( \frac{1}{A} \parallel f \parallel^2 \varepsilon_n^2(f) \right)^{\frac{1}{2}} .$$

By Proposition 1.1(v), we have $\parallel I - R^{N+1} \parallel \leq 1 + q^{N+1}$ ($q = \frac{B-A}{B+A}$). So

$$\parallel r_n^N(f) \parallel \leq \frac{1}{\sqrt{A}} \left(1 + q^{N+1}\right) \parallel f \parallel \varepsilon_n(f).$$

Finally, by (3.3), we obtain the conclusion of Lemma 3.1.

We will approximate to $\sigma_n^N(f)$ by $(\sigma_n^N(f))_m$.

**Lemma 3.3.** Let $\{\varphi_k\}_1^\infty$ be a frame for $H$ with bounds $A,\ B$, and let $\sigma_n^N(f)$ and $(\sigma_n^N(f))_m$ be stated in (2.1) and Definition 2.3, respectively. Then for any $f \in H$, we have

$$\parallel \sigma_n^N(f) - (\sigma_n^N(f))_m \parallel \leq \sqrt{B} \parallel f \parallel \left(1 + \frac{2}{A+B}\right)^{N+1} \sqrt{n}\ \alpha_{N,n,m} \qquad (N,\ n,\ m \in Z^+),$$

where

$$\alpha_{N,n,m} := \max_{\substack{1 \leq l \leq N \\ 1 \leq k \leq n}} \parallel S^l \varphi_k - S_m^l \varphi_k \parallel . \tag{3.6}$$

**Proof.** By (1.5), we have

$$\widetilde{\varphi}_k^N = \widetilde{\varphi}_k^{N-1} + \frac{2}{A+B}\ \varphi_k + \frac{2}{A+B} \sum_{l=1}^{N} \left( \begin{array}{c} N \\ l \end{array} \right) \left( -\frac{2}{A+B} \right)^l S^l \varphi_k.$$

and by (2.4), we have

$$(\widetilde{\varphi}_k^N)_m = (\widetilde{\varphi}_k^{N-1})_m + \frac{2}{A+B}\ \varphi_k + \frac{2}{A+B} \sum_{l=1}^{N} \left( \begin{array}{c} N \\ l \end{array} \right) \left( -\frac{2}{A+B} \right)^l S_m^l \varphi_k \tag{3.7}$$

So we get

$$\widetilde{\varphi}_k^N - (\widetilde{\varphi}_k^N)_m = \widetilde{\varphi}_k^{N-1} - (\widetilde{\varphi}_k^{N-1})_m + \frac{2}{A+B}\sum_{l=1}^{N}\begin{pmatrix} N \\ l \end{pmatrix}\left(-\frac{2}{A+B}\right)^l (S^l\varphi_k - S_m^l\varphi_k).$$

Recursively using the above formula, we have

$$\widetilde{\varphi}_k^N - (\widetilde{\varphi}_k^N)_m = (\widetilde{\varphi}_k^1 - (\widetilde{\varphi}_k^1)_m + \frac{2}{A+B}\sum_{j=2}^{N}\sum_{l=1}^{j}\begin{pmatrix} j \\ l \end{pmatrix}\left(-\frac{2}{A+B}\right)^l (S^l\varphi_k - S_m^l\varphi_k). \tag{3.8}$$

By Definition 2.2 and (1.5), we get

$$\widetilde{\varphi}_k^1 - (\widetilde{\varphi}_k^1)_m = -\left(\frac{2}{A+B}\right)^2 (S\varphi_k - S_m\varphi_k).$$

From this and (3.8), we have

$$\widetilde{\varphi}_k^N - (\widetilde{\varphi}_k^N)_m = \frac{2}{A+B}\sum_{j=1}^{N}\sum_{l=1}^{j}\begin{pmatrix} j \\ l \end{pmatrix}\left(-\frac{2}{A+B}\right)^l (S^l\varphi_k - S_m^l\varphi_k).$$

So we obtain

$$\| \widetilde{\varphi}_k^N - (\widetilde{\varphi}_k^N)_m \| \leq \frac{2}{A+B}\left(\sum_{j=1}^{N}\sum_{l=1}^{j}\begin{pmatrix} j \\ l \end{pmatrix}\left(\frac{2}{A+B}\right)^l\right)\cdot \max_{1\leq l\leq N} \| S^l\varphi_k - S_m^l\varphi_k \| .$$

However,

$$\frac{2}{A+B}\sum_{j=1}^{N}\sum_{l=1}^{j}\begin{pmatrix} j \\ l \end{pmatrix}\left(\frac{2}{A+B}\right)^l \leq \frac{2}{A+B}\sum_{j=1}^{N}\left(1+\frac{2}{A+B}\right)^j \leq \left(1+\frac{2}{A+B}\right)^{N+1}.$$

So we have

$$\| \widetilde{\varphi}_k^N - (\widetilde{\varphi}_k^N)_m \| \leq \left(1+\frac{2}{A+B}\right)^{N+1} \cdot \max_{1\leq l\leq N} \| S^l\varphi_k - S_m^l\varphi_k \| . \tag{3.9}$$

By (2.1) and Definition 2.3, using cauchy's inequality, we obtain that for any $f \in H$,

$$\begin{aligned} \| \sigma_n^N(f) - (\sigma_n^N(f))_m \| &\leq \sum_{k=1}^{n} |(f,\varphi_k)| \cdot \| \widetilde{\varphi}_k^N - (\widetilde{\varphi}_k^N)_m \| \\ &\leq \left(\sum_{k=1}^{n} |(f,\varphi_k)|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{n} \| \widetilde{\varphi}_k^N - (\widetilde{\varphi}_k^N)_m \|^2\right)^{\frac{1}{2}} \end{aligned} \tag{3.10}$$

Combining (3.9)-(3.10) with (3.6), we get

$$\| \sigma_n^N(f) - (\sigma_n^N(f))_m \| \leq \left(1+\frac{2}{A+B}\right)^{N+1} \sqrt{n}\, \alpha_{N,n,m} \left(\sum_{k=1}^{n} |(f,\varphi_k)|^2\right)^{\frac{1}{2}}$$

From this and (1.1), we get the conclusion of Lemma 3.3.

Since

$$\| f - (\sigma_n^N(f))_m \| \leq \| f - \sigma_n^N(f) \| + \| \sigma_n^N(f) - (\sigma_n^N(f))_m \|,$$

by Lemma 3.1 and Lemma 3.3, we conclude immediately the following

**Theorem 3.4.** Let $\{\varphi_k\}_1^\infty$ be a frame with bounds $A$ and $B$, and let

$$(\sigma_n^N(f))_m = \sum_{k=1}^N (f, \varphi_k)(\widetilde{\varphi}_k^N)_m \quad (f \in H),$$

where $(\widetilde{\varphi}_k^N)_m$ is stated in Definition 2.2. Then for any $f \in H$ and $N,\ n,\ m \in Z^+$, we have

$$\begin{aligned} \| f - (\sigma_n^N(f))_m \| \quad &\leq \quad q^{N+1} \| f \| \\ &+ \quad \tfrac{1}{\sqrt{A}} \left(1 + q^{N+1}\right) \| f \| \varepsilon_n(f) + \sqrt{B} \| f \| \left(1 + \tfrac{2}{A+B}\right)^{N+1} \sqrt{n}\ \alpha_{N,n,m}, \end{aligned} \tag{3.11}$$

where $\varepsilon_n(f)$ is stated in (3.1), $q = \frac{B-A}{B+A}$, and

$$\alpha_{N,n,m} = \max_{\substack{1 \leq l \leq N \\ 1 \leq k \leq n}} \| S^l \varphi_k - S_m^l \varphi_k \| \quad (S \text{ is the frame operator}).$$

**Remark 3.5.** In Lemma 2.5, we have shown that $(\sigma_n^N(f))_m$ is a linear combination of $\varphi_1, ..., \varphi_\lambda$ ($\lambda = \max\{m,\ n\}$). So Theorem 3.4 gives a formula approximating to any $f \in H$ by a linear combination of finitely many frame elements.

**Remark 3.6.** Denote

$$\begin{aligned} R_1 &:= q^{N+1} \| f \|, \\ R_2 &:= \frac{1}{\sqrt{A}} \left(1 + q^{N+1}\right) \| f \| \varepsilon_n(f), \\ R_3 &:= \sqrt{B} \| f \| \left(1 + \frac{2}{A+B}\right)^{N+1} \sqrt{n}\ \alpha_{N,n,m}. \end{aligned} \tag{3.12}$$

Since $q = \frac{B-A}{B+A} < 1$, we see that $R_1 \to 0$ ($N \to \infty$). By Remark 3.2, we have $R_2 \to 0$ ($n \to \infty$). From Lemma 2.4(ii) and (3.6), we obtain $\alpha_{N,n,m} \to 0$ ($m \to \infty$), so we have $R_3 \to 0$ ($m \to \infty$).

Therefore, for any $f \in H$ and an approximation error $\varepsilon > 0$, first we choose $N$ such that $R_1 < \frac{\varepsilon}{3}$, next, we choose $n$ such that $R_2 < \frac{\varepsilon}{3}$, finally, for fixed $N,\ n$, we choose $m$ such that $R_3 < \frac{\varepsilon}{3}$. Then from (3.11), we have $\| f - (\sigma_n^N(f))_m \| < \varepsilon$.

Our result is a generalization of a known result on the orthonormal bases[2].

**Remark 3.7.** When $\{\varphi_k\}$ is an orthonormal basis for $H$, the frame bounds $B = A = 1$ and the frame operator $S$ and $S_m$ are both the identity operator. By (3.6),

$$\alpha_{N,n,m} = 0.$$

By (1.4), (1.5), (2.3), and (3.12), we have

$$\widetilde{\varphi}_k = \widetilde{\varphi}_k^N = (\widetilde{\varphi}_k^N)_m = \varphi_k \qquad \text{and} \qquad R_1 = R_3 = 0, \quad R_2 = \left( \sum_{k=n+1}^{\infty} |(f, \varphi_k)|^2 \right)^{\frac{1}{2}}.$$

Therefore, Theorem 3.4 is reduced to a well-known result in Hilbert space[2].

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