# On an efficient sparse representation of objects of general shape via continuous extension and wavelet approximation * 

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#### Abstract

In this paper, we discuss the continuous extension and wavelet approximation of the detected object on a general domain $\Omega$ of $\mathbb{R}^{2}$. We first extend continuously the image to a square $T$ and such that it vanishes on the boundary $\partial T$. On $T \backslash \Omega$, the extension has a simple and clear representation which is determined by the equation of the boundary $\partial \Omega$. We expand the extension into wavelet series on $\mathbb{R}^{2}$. Since the extension tool is polynomials, by the moment theorem, we know that the sequence of wavelet coefficients obtained by us is sparse. Therefore, we can approximate and analyze the internal information of the object very well even if we only store a few wavelet coefficients.


Key words: object-oriented image analysis, continuous extension, wavelet approximation

## 1 Introduction

In the field of image analysis, detection and segmentation of objects of interest in a given image have drawn considerable attention of both researchers and practitioners for more than 30 years. There have been significant progress in this area since late 1980s due to the deeper theoretical understanding of the problems and the rapid increase of computational power. The examples of such progress include: "snakes" [8],"balloons" [9], geodesic active contours [10], geodesic active regions [11], optimal edge integrators [12], the variational method of Mumford and Shah [13], the MDL-based image partitioning method [14], to name a few. One is interested in analyzing the objects after their boundaries are detected and they are segmented either manually by a human interpreter using a pointing device or automatically by one or more of the methods listed above. One needs a set of tools to manipulate the detected objects, for example, to compress them, catalog them, analyze their characteristics, and filter their certain spatial frequency features in an individual manner. Although it is true that the boundary of the object provides us with important geometric and shape information, it becomes a

[^0]nuisance, particularly for the analysis of the internal information (e.g., textures) of the object. For example, if we embed such an object in an otherwise empty rectangle and expand it into the Fourier series in a bruteforce manner, the boundary of the object generates the infamous Gibbs' phenomenon, which prevents us from effectively analyzing the spatial frequency information inside of the object. In order to analyze the object locally, Saito, Yamatani, and Zhao [6][7, Chap.4] have generalized polyharmonic local sine transform in [2] and used it to decouple the boundary/ geometry information from the internal information. Their idea is to split an object of general shape into two components. The first component is a polyharmonic function on the general domain which is determined by the values of the image on the boundary. The second component can be extended smoothly to a square, and then can be approximated by Fourier series.

In this paper, we will present a new algorithm to deal with objects of general shape. We will embed an object of general shape in a square, extend directly it from a general domain to a square, and then we approximate the extension by wavelet series. In detail, we consider an object $f$ living on a general domain $\Omega$. First we continuously extend $f$ from $\Omega$ to a square $T$ such that the extension $F \in l i p 1$ on $T$ and vanishes on the boundary of square $T$. In our extension, we use some simple polynomials as tool. We divide the complement of the domain $\Omega$ in the square into some trapezoids with a curved side and some rectangles. On each trapezoid, the extension $F$ is a polynomial with respect to a variable. On each rectangle, the extension $F$ is a polynomial of two variables. Finally, we approximate $F$ by wavelet series. The obtained wavelet coefficients are sparse, thanks to the moment theorem of wavelets. This enables us to see that the algorithm presented by us can represent data efficiently. If we do not continuously extend $f$ and simply take $f=0$ on the exterior of $\Omega$, since the discontinuity of $f$ at the boundary of $\Omega$, the wavelet coefficients decay very slow. On the other hand, we know that it is very difficult to construct wavelet basis on a general domain, we cannot expect to realize the wavelet approximation by a wavelet basis on a general domain. So our algorithm is a good tool.

Comparing with the algorithm in [6][7, Chap.4], the advantages of our algorithm lie in: (a) We do not need to store the values of the object on the boundary. (b) With the help of the moment theorem of wavelet,
most of wavelet coefficients in our algorithm are zero, while none of Fourier coefficients in the algorithm in [6][7, Chap.4] are zero. (c) The decay of wavelet coefficients depends on the local smoothness, while the decay of Fourier coefficients depends on the whole smoothness, so the decay of wavelet coefficients is faster.

This paper is organized as follows. In Section 2, we discuss a partition of the complement of a general domain in a large square. We will divide this complement into some trapezoids with a curved side and some rectangles. In this section, we also recall some concepts and known results of wavelets. In Section 3, we state the main results. In Section 4 we present our extension algorithm. In Section 5, we show that the sequence of wavelet coefficients of the extension is sparse and wavelet coefficients decay fast. In Section 6, we give a numerical experiment to explain our theory.

## 2 Preliminary

In this section, we discuss partitions of complements of domains in a large square and recall some results about wavelets.

### 2.1 Partition of the complement of the domain

Let $T=[0,1]^{2}$ and $\Omega \subset T$ be a domain. Without loss of generality, we can divide the closed domain $\mathbb{T} \backslash \Omega$ into some rectangles and trapezoids with a curved side. For convenience of representation, we assume that we can choose four points $\left(x_{i}, y_{i}\right) \in \partial \Omega(i=1,2,3,4)$ such that $T \backslash \Omega$ can be divided into the four rectangles (as Fig. 1)

$$
\begin{align*}
& H_{1}=\left[0, x_{1}\right] \times\left[0, y_{1}\right], \quad H_{2}=\left[x_{2}, 1\right] \times\left[0, y_{2}\right], \\
& H_{3}=\left[x_{3}, 1\right] \times\left[y_{3}, 1\right], \quad H_{4}=\left[0, x_{4}\right] \times\left[y_{4}, 1\right] \tag{2.1}
\end{align*}
$$

and four trapezoids with a curved side (as Fig. 1)

$$
E_{1}=\left\{(x, y) ; \quad x_{1} \leq x \leq x_{2}, \quad 0 \leq y \leq g(x)\right\}
$$



Fig.1. Partition of the complement of the domain $\Omega$

$$
\begin{align*}
& E_{2}=\left\{(x, y) ; \quad h(y) \leq x \leq 1, \quad y_{2} \leq y \leq y_{3}\right\} \\
& E_{3}=\left\{(x, y) ; \quad x_{4} \leq x \leq x_{3}, \quad g^{*}(x) \leq y \leq 1\right\} \\
& E_{4}=\left\{(x, y) ; \quad 0 \leq x \leq h^{*}(y), \quad y_{1} \leq y \leq y_{4}\right\} \tag{2.2}
\end{align*}
$$

where $g, h, g^{*}, h^{*} \in C^{1}$. Denote

$$
\begin{aligned}
& M=\max \left\{g(x), \quad x_{1} \leq x \leq x_{2}\right\} ; \quad \tau=\min \left\{g(x), \quad x_{1} \leq x \leq x_{2}\right\} \\
& M^{*}=\max \left\{g^{*}(x), \quad x_{4} \leq x \leq x_{3}\right\} ; \quad \tau^{*}=\min \left\{g^{*}(x), \quad x_{4} \leq x \leq x_{3}\right\}
\end{aligned}
$$

and $0<\tau \leq y_{1}, y_{2} \leq M<M^{*}, \tau \leq \tau^{*} \leq y_{3}, y_{4} \leq M^{*}<1$. Denote

$$
\begin{array}{ll}
N=\max \left\{h(y), \quad y_{2} \leq y \leq y_{3}\right\} ; & \lambda=\min \left\{h(y), \quad y_{2} \leq y \leq y_{3}\right\} \\
N^{*}=\max \left\{h^{*}(y), \quad y_{1} \leq y \leq y_{4}\right\} ; & \lambda^{*}=\min \left\{h^{*}(y), \quad y_{1} \leq y \leq y_{4}\right\}
\end{array}
$$

and $0<\lambda^{*} \leq x_{1}, x_{4} \leq N^{*}<N, \quad \lambda \leq x_{2}, x_{3} \leq N<1$.
From the above construction, we know that $T$ can be expressed into a disjoint union as follows.

$$
\begin{equation*}
T=\Omega \bigcup\left(\bigcup_{i=1}^{4} E_{i}\right) \bigcup\left(\bigcup_{i=1}^{4} H_{i}\right) \tag{2.3}
\end{equation*}
$$

### 2.2 Tensor product wavelets

Now we recall some concepts and known results of wavelets $[1,5,15,16]$.
Let $\psi \in L^{2}(\mathbb{R})$. Its integral translates and dyadic dilations denoted by $\psi_{m, n}$, i.e.,

$$
\psi_{m, n}=2^{\frac{m}{2}} \psi\left(2^{m} \cdot-n\right), \quad m, n \in \mathbb{Z}
$$

We say $\psi$ is an orthonormal wavelet of $L^{2}(\mathbb{R})$ if $\left\{\psi_{m, n}\right\}_{m, n \in \mathbb{Z}}$ is an orthonormal basis of $L^{2}(\mathbb{R})$. We say $\psi$ is compactly supported if its support $D=\operatorname{clos}\{x \in \mathbb{R}, \psi(x) \neq 0\}$ is a compact set on $\mathbb{R}$, write $D=\operatorname{supp} \psi$.

It is well-known that the ordinary tensor product wavelet $\left\{\psi^{(1)}, \psi^{(2)}, \psi^{(3)}\right\}$ generated by a scaling function $\varphi$ and its corresponding wavelet $\psi$, where

$$
\begin{equation*}
\psi^{(1)}(x, y)=\varphi(x) \psi(y), \quad \psi^{(2)}(x, y)=\psi(x) \varphi(y), \quad \psi^{(3)}(x, y)=\psi(x) \psi(y) \tag{2.4}
\end{equation*}
$$

and so $\left\{\psi_{m, n}^{(\nu)}(x, y)\right\}, m \in \mathbb{Z}, n \in \mathbb{Z}^{2}, \nu=1,2,3$ is an ordinary tensor product wavelet basis.
By (2.4), we have

$$
\begin{array}{lr}
\psi_{m, n}^{(1)}(x, y)=\varphi_{m, n_{1}}(x) \psi_{m, n_{2}}(y), & \psi_{m, n}^{(2)}(x, y)=\psi_{m, n_{1}}(x) \psi_{m, n_{2}}(y) \\
\psi_{m, n}^{(3)}(x, y)=\psi_{m, n_{1}}(x) \psi_{m, n_{2}}(y), & \left(m \in \mathbb{Z} n=\left(n_{1}, n_{2}\right) \in \mathbb{Z}^{2}\right) \tag{2.5}
\end{array}
$$

Any $f \in L^{2}\left(\mathbb{R}^{2}\right)$ can be expanded into the wavelet series with convergence in $L^{2}\left(\mathbb{R}^{2}\right)$-norm

$$
f=\sum_{\nu=1}^{3} \sum_{m, n} c_{m, n}^{(\nu)} \psi_{m, n}^{(\nu)}
$$

where $c_{m, n}^{(\nu)}=\int_{\mathbb{R}^{2}} f(x, y) \overline{\psi_{m, n}^{(\nu)}}(x, y) \mathrm{d} x \mathrm{~d} y$ is called the wavelet coefficient.
In our argument, we need the following moment theorem and a characterization of Holder space by wavelets as well as the relation between the local smoothness of functions and wavelet coefficients. We say a function $f \in C^{s+\alpha}(\Omega)$ if its partial derivatives $\frac{\partial^{i+j} f}{\partial x^{i} \partial y^{j}} \in \operatorname{lip} \alpha(0<\alpha \leq 1)$ on $\Omega$ for $i+j \leq s$.

Proposition $2.1[1]$. Let $\psi \in C^{s}(\mathbb{R})$ be a completely supported wavelet. Then, for each $l=0,1, \ldots, s$, we have $\int_{\mathbb{R}} x^{l} \psi(x) \mathrm{d} x=0$.

Proposition 2.2 [5]. Let $\left\{\psi^{(\nu)}\right\}_{1}^{3} \subset C^{s+2}\left(\mathbb{R}^{2}\right)$ be a completely supported wavelet of $L^{2}\left(\mathbb{R}^{2}\right)$.
(a) If $f \in C^{s}\left(\mathbb{R}^{2}\right)$, then wavelet coefficients satisfy

$$
c_{m, n}^{(\nu)}=\int_{\mathbb{R}^{2}} f(x, y) \overline{\psi_{m, n}^{(\nu)}(x, y)} \mathrm{d} x \mathrm{~d} y=O\left(2^{-m(s+1)}\right), \quad \nu=1,2,3
$$

(b) If $f$ is $s$ th-differentiable at a point $\left(x_{0}, y_{0}\right)$ and if $m \in \mathbb{Z}$ and $n=\left(n_{1}, n_{2}\right) \in \mathbb{Z}^{2}$ are such that the point $\left(\frac{n_{1}}{2^{m}}, \frac{n_{2}}{2^{m}}\right)$ in a neighborhood of $\left(x_{0}, y_{0}\right)$, then

$$
c_{m, n}^{(\nu)}=O\left(2^{-m(s+1)}\right)\left(1+\left|\left(2^{m} x_{0}-n_{1}\right)^{2}+\left(2^{m} y_{0}-n_{2}\right)^{2}\right|^{\frac{s}{2}}\right) \quad \nu=1,2,3 . .
$$

## 3 Main results

Let a smooth function $f$ be defined on a general domain $\Omega$ and $\Omega \subset[0,1]^{2}$. We can extend the function $f$ to $\mathbb{R}^{2}$ such that the extension $F \in \operatorname{lip} 1\left(\mathbb{R}^{2}\right)$ and vanishes on the exterior of the unit square. We expand $F$ into the wavelet series, and then we discuss the decay and sparseness of wavelet coefficients. Our main results are stated as follows.

Theorem 3.1. Let $\mathbb{T}:=[0,1]^{2}$ and $\Omega \subset \mathbb{T}$ be a simply connected Jordan closed domain whose boundary is a piecewise smooth curve. If a target function $f(x, y) \in C^{1}(\Omega)$, then there is a function $F(x, y) \in \operatorname{lip} 1\left(\mathbb{R}^{2}\right)$ satisfying

$$
\begin{align*}
& F(x, y)=f(x, y), \quad(x, y) \in \Omega  \tag{3.1}\\
& F(x, y)=0, \quad(x, y) \in \mathbb{R}^{2} \backslash T \tag{3.2}
\end{align*}
$$

On $T \backslash \Omega, F(x, y)$ can be expressed locally in the forms

$$
\begin{equation*}
\sum_{j=0}^{1} \xi_{j}(x) y^{j}, \quad \text { or } \quad \sum_{j=0}^{1} \eta_{j}(y) x^{j}, \quad \text { or } \quad \sum_{i, j=0}^{1} c_{i j} x^{i} y^{j} \tag{3.3}
\end{equation*}
$$

where the functions $\xi_{j}, \eta_{j} \in \operatorname{lip} 1$ and coefficients $c_{i j}$ 's are constants.

We prove this theorem in Section 4.
Let $\Omega \subset T$ and $f \in C^{1}(\Omega)$. We give a partition of $T \backslash \Omega$ as (2.3) in Section 2.

$$
T=\Omega \bigcup\left(\bigcup_{i=1}^{4} E_{i}\right) \bigcup\left(\bigcup_{i=1}^{4} H_{i}\right)
$$

where each $E_{i}$ is a trapezoid with a curved side and each $H_{i}$ is a rectangle. In Section 4 , we define the values of the extension $F$ on $\left\{E_{i}\right\}$ and $\left\{H_{i}\right\}$ such that $F$ satisfies the conditions of Theorem 3.1.

Suppose that $F$ is stated as above. Now we assume that a scaling function $\varphi \in C^{3}(\mathbb{R})$ is supported in a bounded interval and the corresponding wavelet $\psi \in C^{3}(\mathbb{R})$ is also supported a bounded interval. Take a tensor product wavelet $\left\{\psi^{(\nu)}\right\}_{1}^{3}$ which is stated in (2.4). Denote

$$
\begin{equation*}
d_{m, n}=\operatorname{supp} \varphi_{m, n}, \quad D_{m, n}=\operatorname{supp} \psi_{m, n} \tag{3.4}
\end{equation*}
$$

Clearly, $d_{m, n}$ and $D_{m, n}$ are both bounded intervals. We expand $F$ into the wavelet series with respect to the wavelet basis $\left\{\psi_{m, n}^{(\nu)}\right\}_{1}^{3}$

$$
F(x, y)=\sum_{\nu=1}^{3} \sum_{m, n} c_{m, n}^{(\nu)} \psi_{m, n}^{(\nu)}(x, y)
$$

with convergence in the $L^{2}\left(\mathbb{R}^{2}\right)$-norm, where wavelet coefficients

$$
\begin{equation*}
c_{m, n}^{(\nu)}=\int_{\mathbb{R}^{2}} F(x, y) \overline{\psi_{m, n}^{(\nu)}(x, y)} \mathrm{d} x \mathrm{~d} y, \quad m \in \mathbb{Z}, \quad n \in \mathbb{Z}^{2} \tag{3.5}
\end{equation*}
$$

Then wavelet coefficients decay fast and a lot of wavelet coefficients vanish. Precisely say, we have
Theorem 3.2. Let $F(x, y)$ be an extension from $\Omega$ to $\mathbb{R}^{2}$ as was stated in Theorem 3.1 and let $\left\{c_{m, n}^{(\nu)}\right\}$ be wavelet coefficients of $F$, which are stated in (3.5).
(i) For $m \in \mathbb{Z}, n \in \mathbb{Z}^{2}$, and $\nu=1,2,3$, we have $c_{m, n}^{(\nu)}=O\left(2^{-2 m}\right)$.
(ii) Let $m \in \mathbb{Z}, n=\left(n_{1}, n_{2}\right) \in \mathbb{Z}^{2}$ such that $\left(\frac{n_{1}}{2^{m}}, \frac{n_{2}}{2^{m}}\right)$ is in a neighborhood of the point $\left(x_{0}, y_{0}\right)$. If $\left(x_{0}, y_{0}\right) \notin\left(\Omega \bigcup\left(\bigcup_{i=1}^{4} E_{i}\right)\right)$, then

$$
c_{m, n}^{(\nu)}=O\left(2^{-m l}\right)\left(1+\left(A_{m, n}\left(x_{0}, y_{0}\right)\right)^{\frac{l}{2}}\right)
$$

where $A_{m, n}\left(x_{0}, y_{0}\right)=\left(2^{m} x_{0}-n_{1}\right)^{2}+\left(2^{m} y_{0}-n_{2}\right)^{2}$ and $l$ can be arbitrarily large.

The proof of Theorem 3.2 sees Section 5.1.

The following theorem shows sparseness of wavelet coefficients.
Theorem 3.3. Let $c_{m, n}^{(\nu)}$ be wavelet coefficients of $F$. For $m \in \mathbb{Z}, n=\left(n_{1}, n_{2}\right) \in \mathbb{Z}^{2}$,
(i) If $d_{m, n_{1}} \bigcap\left[\lambda^{*}, N\right]=\emptyset$ or $D_{m, n_{2}} \bigcap\left(\left[\tau, M^{*}\right] \bigcup\{0\} \bigcup\{1\}\right)=\emptyset$, then $c_{m, n}^{(1)}=0$;
(ii) If $D_{m, n_{1}} \bigcap\left(\left[\lambda^{*}, N\right] \bigcup\{0\} \bigcup\{1\}\right)=\emptyset$ or $d_{m, n_{2}} \bigcap\left[\tau, M^{*}\right]=\emptyset$, then $c_{m, n}^{(2)}=0$;
(iii) If $D_{m, n_{1}} \bigcap\left(\left[\lambda^{*}, N\right] \bigcup\{0\} \bigcup\{1\}\right)=\emptyset$ or $D_{m, n_{2}} \bigcap\left(\left[\tau, M^{*}\right] \bigcup\{0\} \bigcup\{1\}\right)=\emptyset$, then $c_{m, n}^{(3)}=0$,
where $D_{m, n}$ and $d_{m, n}$ are stated in (3.4) and where $\tau, M^{*}, \lambda^{*}, N$, and $x_{i}, y_{i}$ are stated in Section 2 .
The proof of this theorem is in Section 5.2.
From Theorem 3.2, we see that our extension method is such that a lot of wavelet coefficients vanish. Besides this we find that our extension is such that more wavelet coefficients are very small. One regards very small wavelet coefficients as zeros in application.

Remark 3.4. For $f \in C^{1}(\Omega)$, if we do not extend continuously $f$ and define simply

$$
F^{(0)}(x, y)= \begin{cases}f(x, y), & (x, y) \in \Omega \\ 0, & (x, y) \in \mathbb{R}^{2} \backslash \Omega\end{cases}
$$

then, for the global estimate of wavelet coefficients, we only have $c_{m, n}=O\left(2^{-m}\right)$. Since the discontinuity of $F^{(0)}(x, y)$ at the boundary of $\Omega$, we know that wavelet coefficients near the boundary decay slowly. On the other hand, we know that the construction of the wavelet basis for $L^{2}(\Omega)$ is very difficult, so the above continuous extension of an image is very useful for wavelet approximation.

## 4 Continuous extension of functions

Let $T=[0,1]^{2}$ and let $\Omega \subset T$ be a simply connected Jordan domain whose boundary is a piecewise smooth curve. We can divide the complement $T \backslash \Omega$ into some rectangles and trapezoids with a curved side. With loss
of generality, we can divide the complement $T \backslash \Omega$ as in Section 2.

$$
T \backslash \Omega=\left(\bigcup_{i=1}^{4} E_{i}\right) \bigcup\left(\bigcup_{i=1}^{4} H_{i}\right)
$$

where $E_{i}$ and $H_{i}$ are stated in (2.1) and (2.2), respectively.
Let $f(x, y) \in C^{1}(\Omega)$. We define $F(x, y)=f(x, y)$ on $\Omega$, and then we will define $F(x, y)$ on each $E_{i}$ and $H_{i}$ such that $F(x, y) \in \operatorname{lip} 1\left(\mathbb{R}^{2}\right)$ and (3.2) holds.

### 4.1 Continuous extension on trapezoids with a curved side

First we continuously extend $f$ from the domain $\Omega$ to each trapezoid $E_{i}$.

For $i=1,2,3,4$, we define $P_{i}(x, y)$ on $E_{i}$ as follows.

$$
\begin{array}{ll}
P_{1}(x, y)=f(x, g(x)) \frac{y}{g(x)}, & (x, y) \in E_{1}, \\
P_{2}(x, y)=f(h(y), y) \frac{1-x}{1-h(y)}, & (x, y) \in E_{2}, \\
P_{3}(x, y)=f\left(x, g^{*}(x)\right) \frac{1-y}{1-g^{*}(x)}, & (x, y) \in E_{3}, \\
P_{4}(x, y)=f\left(h^{*}(y), y\right) \frac{x}{h^{*}(y)}, & (x, y) \in E_{4} . \tag{4.1}
\end{array}
$$

Lemma 4.1. Let $f \in C^{1}(\Omega)$ and

$$
F(x, y)= \begin{cases}f(x, y), & (x, y) \in \Omega  \tag{4.2}\\ P_{i}(x, y), & (x, y) \in E_{i}, \quad i=1,2,3,4\end{cases}
$$

Then

$$
F(x, y) \in l i p 1 \quad \text { on } \quad \Omega \bigcup\left(\bigcup_{i=1}^{4} E_{i}\right) \quad \text { and } \quad F(x, y)=0 \quad \text { on } \quad \partial T \bigcap\left(\bigcup_{i=1}^{4} E_{i}\right) .
$$

Proof. For $(x, y) \in \Omega \bigcap E_{1}$, we have $y=g(x)$. So, by (4.1) and (4.2), we get

$$
F(x, y)=P_{1}(x, y)=f(x, g(x))=f(x, y), \quad(x, y) \in\left(\Omega \bigcap E_{1}\right)
$$

From this, we know that $F(x, y)$ is well defined on $\Omega \bigcup E_{1}$ and $F \in C\left(\Omega \bigcup E_{1}\right)$.

Since $F(x, y)=f(x, y)((x, y) \in \Omega)$ and $f \in C^{1}(\Omega)$, we have $F \in \operatorname{lip} 1(\Omega)$. Since $f \in C^{1}\left(\Omega \bigcap E_{1}\right), g \in C^{1}$, and $g(x)>0\left(x_{1} \leq x \leq x_{2}\right)$, by (4.1), we have $P_{1} \in \operatorname{lip} 1\left(E_{1}\right)$. Again since $F \in C\left(\Omega \bigcup E_{1}\right)$, by (4.2), we finally have $F \in \operatorname{lip} 1\left(\Omega \bigcup E_{1}\right)$. Similarly, we can prove that $F \in \operatorname{lip} 1\left(\Omega \bigcup E_{i}\right)$ for each $i$.

Note that $\partial T \bigcap E_{1}=\left\{(x, y), x_{1} \leq x \leq x_{2}, y=0\right\}$. When $(x, y) \in \partial T \bigcap E_{1}$, we have $y=0$, so $P_{1}(x, y)=0$. By (4.2), we have $F(x, y)=0,(x, y) \in \partial T \bigcap E_{1}$. Similarly, for each $i, F(x, y)=0$ on $\partial T \bigcap E_{i}$. Lemma 4.1 is proved.

### 4.2 Continuous extension to rectangles

Now we continuously extend $F$ from $\Omega \bigcup\left(\bigcup_{i=1}^{4} E_{i}\right)$ to $T$.
For each $i=1,2,3,4$, we define $Q_{i}(x, y)$ on $H_{i}$ as follows:

$$
\begin{array}{ll}
Q_{1}(x, y)=\frac{f\left(x_{1}, y_{1}\right)}{x_{1} y_{1}} x y, & (x, y) \in H_{1} \\
Q_{2}(x, y)=\frac{f\left(x_{2}, y_{2}\right)}{\left(1-x_{2}\right) y_{2}}(1-x) y, & (x, y) \in H_{2} \\
Q_{3}(x, y)=\frac{f\left(x_{3}, y_{3}\right)}{\left(1-x_{3}\right)\left(1-y_{3}\right)}(1-x)(1-y), & (x, y) \in H_{3} \\
Q_{4}(x, y)=\frac{f\left(x_{4}, y_{4}\right)}{x_{4}\left(1-y_{4}\right)} x(1-y), & (x, y) \in H_{4} \tag{4.3}
\end{array}
$$

Lemma 4.2. Let

$$
F(x, y)= \begin{cases}f(x, y), & (x, y) \in \Omega  \tag{4.4}\\ P_{i}(x, y), & (x, y) \in E_{i} \\ Q_{i}(x, y), & (x, y) \in H_{i}, \quad i=1,2,3,4 \\ 0, & (x, y) \in \mathbb{R}^{2} \backslash T\end{cases}
$$

Then $F(x, y) \in l i p 1$ on $\mathbb{R}^{2}$.
Proof. By (2.3), we have $T=\bigcup_{i=1}^{4} G_{i}$, where

$$
G_{1}=H_{1} \bigcup E_{1} \bigcup E_{4} \bigcup \Omega, \quad G_{i}=H_{i} \bigcup E_{i-1} \bigcup E_{i} \bigcup \Omega \quad(i=2,3,4)
$$

We only need to prove $F \in l i p 1$ on each $G_{i}$. First we consider $G_{1}$.
On $E_{1} \bigcap H_{1}$, we have $x=x_{1}$ and $g(x)=y_{1}$, so, by (4.1) and (4.3),

$$
Q_{1}(x, y)=f\left(x_{1}, y_{1}\right) \frac{y}{y_{1}}=P_{1}(x, y)
$$

On $E_{4} \bigcap H_{1}$, we have $y=y_{1}$ and $h^{*}\left(y_{1}\right)=x_{1}$, so

$$
Q_{1}(x, y)=f\left(x_{1}, y_{1}\right) \frac{x}{x_{1}}=P_{4}(x, y)
$$

From this we see that $F$ is well-defined on $E_{1} \bigcup H_{1} \bigcup E_{4}$ and $F \in C\left(E_{1} \bigcup H_{1} \bigcup E_{4}\right)$. By Lemma 4.1, we have $F \in$ $\operatorname{lip} 1\left(E_{1} \bigcup E_{4}\right)$. Again by $F \in \operatorname{lip} 1\left(H_{1}\right)$, we get $F \in \operatorname{lip} 1\left(E_{1} \bigcup H_{1} \bigcup E_{4}\right)$. From this and $F \in \operatorname{lip} 1\left(E_{1} \bigcup \Omega \bigcup E_{4}\right)$ (by Lemma 4.1), we have $F \in \operatorname{lip} 1\left(G_{1}\right)$. Similarly, we have $F \in \operatorname{lip} 1\left(G_{i}\right)(i=2,3,4)$. Therefore, we have $F \in \operatorname{lip} 1(T)$.

We see easily that $F=0$ on $\partial T$. Again noticing that $F(x, y)=0,(x, y) \notin T$, we obtain $F \in \operatorname{lip} 1\left(\mathbb{R}^{2}\right)$. Lemma 4.2 is proved.

Proof of Theorem 3.1. By Lemma 4.2, we know the extension $F \in \operatorname{lip} 1\left(\mathbb{R}^{2}\right)$. From (4.1), (4.3), and (2.3), we see that on $T \backslash \Omega, F(x, y)$ can be expressed locally in the form (3.3). Theorem 3.1 is proved.

## 5 Wavelet approximation

Let $\Omega \subset T=[0,1]^{2}$ and $f \in C^{1}(\Omega)$ be stated in Section 2 . We have extended $f$ continuously to $F \in \operatorname{lip} 1\left(\mathbb{R}^{2}\right)$ in Section 4 such that $F(x, y)=f(x, y)((x, y) \in \Omega)$ and $F(x, y)=0\left((x, y) \in \mathbb{R}^{2} \backslash T\right)$. The representations of $F$ on $T \backslash \Omega$ are stated in Section 4. Now we expand $F$ into a kind of wavelet series.

Let $\varphi \in C^{3}(\mathbb{R})$ be a compactly supported scaling function of $L^{2}(\mathbb{R})$ and $\psi$ be the corresponding wavelet. Assume that the support of $\varphi$ is a closed interval. From this, we know that $\psi \in C^{3}(\mathbb{R})$ and the support of $\psi$ is also a closed interval. Denote $\operatorname{supp} \varphi_{m, n}=d_{m, n}$ and $\operatorname{supp} \psi_{m, n}=D_{m, n}$. Let

$$
\psi^{(1)}(x, y)=\varphi(x) \psi(y), \quad \psi^{(2)}(x, y)=\psi(x) \varphi(y), \quad \psi^{(3)}(x, y)=\psi(x) \psi(y)
$$

So $\left\{\psi^{(1)}, \psi^{(2)}, \psi^{(3)}\right\}$ is an ordinary two-dimensional tensor product wavelet, and so $\left\{\psi_{m, n}^{(1)}, \psi_{m, n}^{(2)}, \psi_{m, n}^{(3)}\right\}$ is a wavelet basis of $L^{2}\left(\mathbb{R}^{2}\right)$. Moreover, $\left\{\psi^{(\nu)}\right\}_{1}^{3} \subset C^{3}\left(\mathbb{R}^{2}\right)$. We expand $F$ into the wavelet series

$$
F(x, y)=\sum_{\nu=1}^{3} \sum_{m, n} c_{m, n}^{(\nu)} \psi_{m, n}^{(\nu)}(x, y)
$$

with convergence in $L^{2}\left(\mathbb{R}^{2}\right)$-norm, where

$$
c_{m, n}^{(\nu)}=\int_{\mathbb{R}^{2}} F(x, y) \overline{\psi_{m, n}^{(\nu)}(x, y)} \mathrm{d} x \mathrm{~d} y=\int_{T} F(x, y) \overline{\psi_{m, n}^{(\nu)}(x, y)} \mathrm{d} x \mathrm{~d} y, m \in \mathbb{Z}, n \in \mathbb{Z}^{2}
$$

since $F(x, y)$ vanishes outside $T$.

### 5.1 Decay of wavelet coefficients

If we do not extend continuously $f$ and define simply

$$
F^{(0)}(x, y)= \begin{cases}f(x, y), & (x, y) \in \Omega \\ 0, & (x, y) \in \mathbb{R}^{2} \backslash \Omega\end{cases}
$$

Then, for wavelet coefficients, we have only the following estimate

$$
\begin{aligned}
\int_{\mathbb{R}^{2}} F^{(0)}(x, y) \overline{\psi_{m, n}^{(1)}(x, y)} \mathrm{d} x \mathrm{~d} y & =2^{m} \int_{\mathbb{R}^{2}} F^{(0)}(x, y) \bar{\varphi}\left(2^{m} x-n\right) \bar{\psi}\left(2^{m} y-n\right) \mathrm{d} x \mathrm{~d} y \\
& =2^{-m} \int_{\mathbb{R}^{2}} F^{(0)}\left(2^{-m}(x+n), 2^{-m}(y+n)\right) \bar{\varphi}(x) \bar{\psi}(y) \mathrm{d} x \mathrm{~d} y \\
& =O\left(2^{-m}\right)
\end{aligned}
$$

Proof of Theorem 3.2. If we continuously extend $f \in C^{1}(\Omega)$ to $F \in \operatorname{lip} 1\left(\mathbb{R}^{2}\right)$ as in Section 4, then, since $F(x, y) \in \operatorname{lip} 1\left(\mathbb{R}^{2}\right)$ and $\left\{\psi^{(\nu)}\right\}_{1}^{3} \subset C^{3}$, by Proposition 2.2 (a), we know that for $\nu=1,2$, 3 , wavelet coefficients have estimates

$$
\begin{equation*}
c_{m, n}^{(\nu)}=\int_{\mathbb{R}^{2}} F(x, y) \overline{\psi_{m, n}^{(\nu)}(x, y)} \mathrm{d} x \mathrm{~d} y=O\left(2^{-2 m}\right) \tag{5.1}
\end{equation*}
$$

We get Theorem 3.2 (i).
By (2.3), we know that

$$
\mathbb{R}^{2}=\Omega \bigcup\left(\bigcup_{i=1}^{4} E_{i}\right) \bigcup\left(\bigcup_{i=1}^{4} H_{i}\right) \bigcup\left(\mathbb{R}^{2} \backslash T\right)
$$

From the construction of $F(x, y)$, we know that $F(x, y)$ possesses the different smoothness in the different domains: $F \in C^{1}(\Omega)$ and $F \in \operatorname{lip} 1$ on each $E_{i}$. For any $l \in \mathbb{Z}_{+}, F \in C^{l}$ on $\mathbb{R}^{2} \backslash T$ and each $H_{i}$. From this, using proposition 2.2 (b), we have Theorem 3.2 (ii).

### 5.2 Sparseness of wavelet coefficients

In construction of $F$, we use polynomials as the tool of construction. It is well known that an orthogonal wavelet is orthonormal to polynomials of lower degree. This enables wavelet coefficients $\left\{c_{m, n}^{(\nu)}\right\}$ to be sparse.

Proof of Theorem 3.3. By (2.3), we know that $T=\Omega \bigcup\left(\bigcup_{i=1}^{4} E_{i}\right) \bigcup\left(\bigcup_{i=1}^{4} H_{i}\right)$. For $m \in \mathbb{Z}, n=\left(n_{1}, n_{2}\right) \in$ $\mathbb{Z}^{2}$, and $\nu=1,2,3$, we have

$$
\begin{align*}
c_{m, n}^{(\nu)} & =\int_{\Omega} F(x, y) \overline{\psi_{m, n}^{(\nu)}}(x, y) \mathrm{d} x \mathrm{~d} y \\
& +\sum_{i=1}^{4} \int_{E_{i}} F(x, y) \overline{\psi_{m, n}^{(\nu)}}(x, y) \mathrm{d} x \mathrm{~d} y+\sum_{i=1}^{4} \int_{H_{i}} F(x, y) \overline{\psi_{m, n}^{(\nu)}}(x, y) \mathrm{d} x \mathrm{~d} y  \tag{5.2}\\
& =I_{1}^{(\nu)}+I_{2}^{(\nu)}+I_{3}^{(\nu)}=I_{1}^{(\nu)}+\sum_{i=1}^{4} I_{2, i}^{(\nu)}+\sum_{i=1}^{4} I_{3, i}^{(\nu)}
\end{align*}
$$

Now we discuss $c_{m, n}^{(1)}$. By (5.2), $c_{m, n}^{(1)}=I_{1}^{(1)}+I_{2}^{(1)}+I_{3}^{(1)}$.
(i) We first consider $I_{2}^{(1)}=\sum_{i=1}^{4} I_{2, i}^{(1)}$.

By (5.2) and (2.5), we have

$$
I_{2, i}^{(1)}=\int_{E_{i}} F(x, y) \overline{\psi_{m, n}^{(1)}}(x, y) \mathrm{d} x \mathrm{~d} y=\int_{E_{i}} F(x, y) \overline{\varphi_{m, n_{1}}}(x) \overline{\psi_{m, n_{2}}}(y) \mathrm{d} x \mathrm{~d} y
$$

For $i=1$, since $E_{1}=\left\{(x, y): x_{1} \leq x \leq x_{2}, 0 \leq y \leq g(x)\right\}$ and (4.1), and (4.4),

$$
F(x, y)=P_{1}(x, y)=f(x, g(x)) \frac{y}{g(x)}, \quad(x, y) \in E_{1}
$$

we get

$$
\begin{equation*}
I_{2,1}^{(1)}=\int_{x_{1}}^{x_{2}} f(x, g(x)) \frac{\overline{\varphi_{m, n_{1}}}(x)}{g(x)}\left(\int_{0}^{g(x)} y \overline{\psi_{m, n_{2}}}(y) \mathrm{d} y\right) \mathrm{d} x . \tag{5.3}
\end{equation*}
$$

By the assumption,

$$
\max _{x_{1} \leq x \leq x_{2}} g(x)=M \quad \text { and } \quad \min _{x_{1} \leq x \leq x_{2}} g(x)=\tau, \quad \text { and } \quad 0<\tau \leq M<1
$$

When $D_{m, n_{2}}=\operatorname{supp} \psi_{m, n_{2}} \subset \mathbb{R} \backslash[0, M]$, we know that in (5.3), the integral

$$
\int_{0}^{g(x)} y \overline{\psi_{m, n_{2}}}(y) \mathrm{d} y=0\left(x_{1} \leq x \leq x_{2}\right) .
$$

When $D_{m, n_{2}} \subset[0, \tau]$. The integral range $[0, g(x)] \supset[0, \tau] \supset D_{m, n_{2}}$. So $\psi_{m, n}(y)=0$ if $y \leq 0$ or $y \geq g(x)$.

Furthermore,

$$
\int_{0}^{g(x)} y \overline{\psi_{m, n_{2}}}(y) \mathrm{d} y=\int_{\mathbb{R}} y \overline{\psi_{m, n_{2}}}(y) \mathrm{d} y .
$$

We compute the integral

$$
\int_{\mathbb{R}} y \overline{\psi_{m, n_{2}}}(y) \mathrm{d} y=2^{\frac{m}{2}} \int_{\mathbb{R}} y \overline{\psi\left(2^{m} y-n_{2}\right)} \mathrm{d} y=2^{-\frac{3 m}{2}} \int_{\mathbb{R}}\left(y+n_{2}\right) \overline{\psi(y)} \mathrm{d} y
$$

Since $\psi \in C^{3}(\mathbb{R})$, by Proposition 2.1, we have $\int_{\mathbb{R}} y^{k} \psi(y) \mathrm{d} y=0(k=0,1)$. So $\int_{0}^{g(x)} y \overline{\psi_{m, n_{2}}}(y) \mathrm{dy}=0$.
Hence, by (5.3), we know that if $D_{m, n_{2}} \bigcap[0, M]=\emptyset$ or $D_{m, n_{2}} \subset[0, \tau]$, we have $I_{2,1}^{(1)}=0$. Again, since $D_{m, n_{2}}$ is an interval, we can conclude that

$$
\begin{equation*}
I_{2,1}^{(1)}=0 \quad \text { if } \quad D_{m, n_{2}} \bigcap\{[\tau, M], 0\}=\emptyset \tag{5.4}
\end{equation*}
$$

Similarly, we have

$$
\begin{equation*}
I_{2,3}^{(1)}=0 \quad \text { if } \quad D_{m, n_{2}} \bigcap\left\{\left[\tau^{*}, M^{*}\right], 1\right\}=\emptyset . \tag{5.5}
\end{equation*}
$$

Noticing that the representations (2.2) of $E_{2}$, by (4.1) and (4.4), we have

$$
I_{2,2}^{(1)}=\int_{y_{2}}^{y_{3}} f(h(y), y) \frac{1}{1-h(y)} \overline{\psi_{m, n_{2}}}(y)\left(\int_{h(y)}^{1}(1-x) \overline{\varphi_{m, n_{1}}}(x) \mathrm{d} x\right) \mathrm{d} y
$$

If $D_{m, n_{2}}=\operatorname{supp} \psi_{m, n_{2}} \subset \mathbb{R} \backslash\left[y_{2}, y_{3}\right]$, then $I_{2,2}^{(1)}=0$. By the assumption,

$$
\max _{y_{2} \leq y \leq y_{3}} h(y)=N, \quad \min _{y_{2} \leq y \leq y_{3}} h(y)=\lambda>0 .
$$

Hence, we have

$$
\begin{equation*}
I_{2,2}^{(1)}=0 \quad \text { if } \quad D_{m, n_{2}} \bigcap\left[y_{2}, y_{3}\right]=\emptyset \tag{5.6}
\end{equation*}
$$

Similarly, we have

$$
\begin{equation*}
I_{2,4}^{(1)}=0 \quad \text { if } \quad D_{m, n_{2}} \bigcap\left[y_{1}, y_{4}\right]=\emptyset \tag{5.7}
\end{equation*}
$$

(ii) From (5.2), we know that

$$
I_{3}^{(1)}=\sum_{i=1}^{4} I_{3, i}^{(1)}, \quad \text { where } \quad I_{3, i}^{(1)}=\int_{H_{i}} F(x, y) \overline{\varphi_{m, n_{1}}}(x) \overline{\psi_{m, n_{2}}}(y) \mathrm{d} x \mathrm{~d} y
$$

where the representation of each square $H_{i}$ is stated in (2.1). By (4.3) and (4.4), we have

$$
F(x, y)=\sum_{k, j=0}^{1} c_{k j}^{(i)} x^{k} y^{j}, \quad(x, y) \in H_{i}
$$

where $c_{k j}^{(i)}$ are constants, so we have

$$
I_{3,1}^{(1)}=\sum_{k, j=0}^{1} c_{k j}^{(1)} \int_{0}^{x_{1}} x^{k} \overline{\varphi_{m, n_{1}}}(x) \mathrm{d} x \int_{0}^{y_{1}} y^{j} \overline{\psi_{m, n_{2}}}(y) \mathrm{d} y
$$

If $D_{m, n_{2}} \subset\left[0, y_{1}\right]$, then we have $\int_{0}^{y_{1}} y^{j} \overline{\psi_{m, n_{2}}}(y) \mathrm{d} y=\int_{\mathbb{R}} y^{j} \overline{\psi_{m, n_{2}}}(y) \mathrm{d} y$. By Proposition 2.1, we can conclude that $\int_{0}^{y_{1}} y^{j} \overline{\psi_{m, n_{2}}}(y) \mathrm{d} y=0$, so $I_{3,1}^{(1)}=0$. If $D_{m, n_{2}} \bigcap\left[0, y_{1}\right]=\emptyset$, then we have $I_{3,1}^{(1)}=0$. Noticing that $D_{m, n_{2}}$ is an interval, we obtain that

$$
\begin{equation*}
I_{3,1}^{(1)}=0 \quad \text { if } \quad D_{m, n_{2}} \bigcap\left(\{0\},\left\{y_{1}\right\}\right)=\emptyset \tag{5.8}
\end{equation*}
$$

Similarly, we have

$$
\begin{array}{ll}
I_{3,2}^{(1)}=0 & \text { if } \\
I_{m, n_{2}}^{(1)}=0 & \text { if } \\
D_{m, n_{2}} \bigcap\left(\left\{y_{3}\right\},\{1\}\right)=\emptyset  \tag{5.9}\\
I_{3,4}^{(1)}=0 & \text { if }
\end{array}
$$

(iii) We discuss $I_{1}^{(1)}$.

By (5.2), we have

$$
I_{1}^{(1)}=\int_{\Omega} F(x, y) \overline{\varphi_{m, n_{1}}}(x, y) \overline{\psi_{m, n_{2}}}(x, y) \mathrm{d} x \mathrm{~d} y
$$

So we deduce that

$$
\begin{equation*}
I_{1}^{(1)}=0 \quad \text { if } \quad\left(d_{m, n_{1}} \times D_{m, n_{2}}\right) \bigcap \Omega=\emptyset \tag{5.10}
\end{equation*}
$$

Below combining (5.4)-(5.10), we give a condition for which $c_{m, n}^{(1)}=0$.
From the partition given in Section 2, we know that $0<\tau \leq y_{1}, y_{2} \leq M$ and $\tau^{*} \leq y_{3}, y_{4} \leq M^{*}$. So

$$
[\tau, M] \bigcup\left[\tau^{*}, M^{*}\right] \bigcup\left[y_{2}, y_{3}\right] \bigcup\left[y_{1}, y_{4}\right]=\left[\tau, M^{*}\right]
$$

From this and (5.4)-(5.7), we know that if $D_{m, n_{2}} \bigcap\left(\left[\tau, M^{*}\right] \bigcup\{0\} \bigcup\{1\}\right)=\emptyset$, then $I_{2}^{(1)}=0$. By (5.8) and (5.9), if $D_{m, n} \bigcap\left\{0,1, y_{1}, y_{2}, y_{3}, y_{4}\right\}=\emptyset$, then we have $I_{3}^{(1)}=0$.

Since $\left\{y_{i}\right\}_{1}^{4} \subset\left[\tau, M^{*}\right]$, it follows that

$$
\begin{equation*}
I_{2}^{(1)}=0 \quad \text { and } \quad I_{3}^{(1)}=0 \quad \text { if } \quad D_{m, n_{2}} \bigcap\left(\left[\tau, M^{*}\right] \bigcup\{0\} \bigcup\{1\}\right)=\emptyset \tag{5.11}
\end{equation*}
$$

From the partition given in Section 2, we know that $\Omega \subset\left[\tau, M^{*}\right] \times\left[\lambda^{*}, N\right]$. Hence, by (5.10), we obtain that

$$
I_{1}^{(1)}=0 \quad \text { if } \quad d_{m, n_{1}} \bigcap\left[\lambda^{*}, N\right]=\emptyset \quad \text { or } \quad D_{m, n_{2}} \bigcap\left[\tau, M^{*}\right]=\emptyset
$$

Again by (5.11), we finally deduce that if $m \in \mathbb{Z}, n \in \mathbb{Z}^{2}$ such that

$$
D_{m, n_{2}} \bigcap\left(\left[\tau, M^{*}\right] \bigcup\{0\} \bigcup\{1\}\right)=\emptyset \quad \text { or } \quad d_{m, n_{1}} \bigcap\left[\lambda^{*}, N\right]=\emptyset
$$

we know that $I_{1}^{(1)}=0, I_{2}^{(1)}=0$, and $I_{3}^{(1)}=0$ simultaneously. So we have

$$
c_{m, n}^{(1)}=I_{1}^{(1)}+I_{2}^{(1)}+I_{3}^{(1)}=0
$$

Theorem 3.3 (i) holds. The proof of Theorem 3.3 (ii) is similar.
Finally, we consider $c_{m, n}^{(3)}=I_{1}^{(3)}+I_{2}^{(3)}+I_{3}^{(3)}$.
At first we discuss $I_{2}^{(3)}=\sum_{i=1}^{4} I_{2, i}^{(3)}$. By (5.2),

$$
\begin{equation*}
I_{2,1}^{(3)}=\int_{x_{1}}^{x_{2}} f(x, g(x)) \frac{\overline{\psi_{m, n_{1}}}(x)}{g(x)}\left(\int_{0}^{g(x)} y \overline{\psi_{m, n_{2}}}(y) \mathrm{d} y\right) \mathrm{d} x \tag{5.12}
\end{equation*}
$$

Similar to the argument of (5.4), we have

$$
I_{2,1}^{(3)}=0 \quad \text { if } \quad D_{m, n_{2}} \bigcap([\tau, M] \bigcup\{0\})=\emptyset .
$$

On the other hand, by (5.12), we have also $I_{2,1}^{(3)}=0$ if $D_{m, n_{1}} \bigcap\left[x_{1}, x_{2}\right]=\emptyset$. So

$$
\begin{equation*}
I_{2,1}^{(3)}=0 \quad \text { if } \quad D_{m, n_{1}} \bigcap\left[x_{1}, x_{2}\right]=\emptyset \quad \text { or } \quad D_{m, n_{2}} \bigcap([\tau, M] \bigcup\{0\})=\emptyset . \tag{5.13}
\end{equation*}
$$

Similarly, we have

$$
\begin{array}{lllll}
I_{2,3}^{(3)}=0 & \text { if } & D_{m, n_{1}} \bigcap\left[x_{4}, x_{3}\right]=\emptyset & \text { or } & D_{m, n_{2}} \bigcap\left(\left[\tau^{*}, M^{*}\right] \bigcup\{1\}\right)=\emptyset . \\
I_{2,2}^{(3)}=0 & \text { if } \quad D_{m, n_{1}} \bigcap([\lambda, N] \bigcup\{1\})=\emptyset & \text { or } & D_{m, n_{2}} \bigcap\left[y_{2}, y_{3}\right]=\emptyset . \\
I_{2,4}^{(3)}=0 & \text { if } & D_{m, n_{1}} \bigcap\left(\left[\lambda^{*}, N^{*}\right] \bigcup\{0\}\right)=\emptyset & \text { or } & D_{m, n_{2}} \bigcap\left[y_{1}, y_{4}\right]=\emptyset . \tag{5.16}
\end{array}
$$

Summarizing these results (5.13)-(5.16), noticing that $I_{2}^{(3)}=\sum_{i=1}^{4} I_{2, i}^{(3)}$, by

$$
\begin{aligned}
& {[\tau, M] \bigcup\left[\tau^{*}, M^{*}\right] \bigcup\left[y_{2}, y_{3}\right] \bigcup\left[y_{1}, y_{4}\right]=\left[\tau, M^{*}\right]} \\
& {[\lambda, N] \bigcup\left[\lambda^{*}, N^{*}\right] \bigcup\left[x_{1}, x_{2}\right] \bigcup\left[x_{4}, x_{3}\right]=\left[\lambda^{*}, N\right]}
\end{aligned}
$$

we have

$$
\begin{equation*}
I_{2}^{(3)}=0 \quad \text { if } \quad D_{m, n_{1}} \bigcap\left(\left[\lambda^{*}, N\right] \bigcup\{0\} \bigcup\{1\}\right)=\emptyset \quad \text { or } \quad D_{m, n_{2}} \bigcap\left(\left[\tau, M^{*}\right] \bigcup\{0\} \bigcup\{1\}\right)=\emptyset . \tag{5.17}
\end{equation*}
$$

Similarly, we have

$$
\begin{gather*}
I_{3}^{(3)}=0 \quad \text { if } \quad D_{m, n_{1}} \bigcap\left\{0,1, x_{1}, x_{2}, x_{3}, x_{4}\right\}=\emptyset \quad \text { or } \quad D_{m, n_{2}} \bigcap\left\{0,1, y_{1}, y_{2}, y_{3}, y_{4}\right\}=\emptyset .  \tag{5.18}\\
I_{1}^{(3)}=0 \quad \text { if } \quad D_{m, n_{1}} \bigcap\left[\lambda^{*}, N\right]=\emptyset \quad \text { or } \quad D_{m, n_{2}} \bigcap\left[\tau, M^{*}\right]=\emptyset . \tag{5.19}
\end{gather*}
$$

Noticing that $c_{m, n}^{(3)}=I_{1}^{(3)}+I_{2}^{(3)}+I_{3}^{(3)}$, combining (5.17)-(5.19), we obtain that for $n=\left(n_{1}, n_{2}\right)$,

$$
c_{m, n}^{(3)}=0 \quad \text { if } \quad D_{m, n_{1}} \bigcap\left(\left[\lambda^{*}, N\right] \bigcup\{0\} \bigcup\{1\}\right)=\emptyset \quad \text { or } \quad D_{m, n_{2}} \bigcap\left(\left[\tau, M^{*}\right] \bigcup\{0\} \bigcup\{1\}\right)=\emptyset
$$

since each $x_{i} \in\left[\lambda^{*}, N\right]$ and each $y_{i} \in\left[\tau, M^{*}\right]$. Theorem 3.3 (iii) holds. Theorem 3.3 is proved.
From (5.2), we see that only if each $I_{1}^{(\nu)}, I_{2, i}^{(\nu)}$, and $I_{3, i}^{(\nu)}$ vanishes $(\mathrm{i}=1,2,3,4)$, we have $c_{m, n}^{(\nu)}=0$. In reality, more wavelet coefficients are very small which are also regarded as zero in application. Since wavelet coefficients decay fast and wavelet coefficients are sparse after the image is continuously extended, the wavelet approximation can compress data very well.

## 6 Image Representation

In this section, we will examine the approximation performance of our extension algorithm. The quality of approximation in this paper is measured by PSNR (or peak signal-to-noise ratio)

$$
\operatorname{PSNR}:=20 \times \log _{10}\left(\max _{x \in \Omega}|f(x)| / \mathrm{RMSE}\right),
$$

where RMSE is the absolute $\ell^{2}$ error between the original and the approximation divided by the square root of the total number of pixels in the original image and $\Omega$ is the support of the original image. The unit of PSNR is decibel ( dB ).

We will use our algorithm to approximate the face part of the Barbara image as shown in Figure 2(a). This is an image on a general domain. We denote this general domain by $\Omega$. The number of the samples of this image on $\Omega$ is 8159 . We extend this image continuously and then approximated the extended image by Coiflet with two vanishing moments. We will compare our algorithm with 2D Coiflet and tensor product of 1D Coiflet. Figure 2 shows our extension process. Figure 3 shows the quality of approximations of Barbara face image using $0-1000$ wavelet coefficients of 2D Coiflet, tensor product of 1D coiflet and our algorithm, respectively. Figure 4 shows reconstructed images on $\Omega$ using top 400 wavelet coefficients of 2D Coiflet, tensor product of 1D coiflet and our algorithm, respectively. We can see that our algorithm can compress the image on a general domain much better that 2D Coiflet and tensor product of 1D Coiflet.

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(a)

(c)

(b)

(d)


Fig.2. Extension of the Barbara face image


Fig.3. PSNR values of the Barbara face image approximated by $0-1000$ wavelet coefficients of 2 D Coiflet, tensor product of 1D Coiflet and our algorithm, respectively


Fig.4. Reconstructed images on $\Omega$ using top 400 wavelet coefficients


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