Highlights

Quantum mechanics can be obtained as a classical theory with a restricted set of observables.
Generalizations of QM can be obtained in similar way. They can be described as geometric theories (theories where the starting point is the set of states). Deterministic theories.
Decoherence can be proved in general geometric theories placed in random environment. One can derive probabilities from decoherence.
Calculations in geometric approach are as easy as in conventional approach (and sometimes easier).
L-functionals. Applications to infrared problem in QED.

Theories with translation symmetry=quantum field theories without fields. Asymptotic commutativity.
In theories with translation symmetry one can define particles and quasiparticles.
If we have asymptotic commutativity or cluster property we can define inclusive scattering matrix.
Inclusive scattering matrix can be expressed in terms of Keldysh Green's functions on shell.
The notion of inclusive scattering matrix can be introduced in geometric theories.

Cross-section =probability density of the process $(A, B) \to (M, N, ..., R)$
Inclusive cross-section=probability density of the process $(A, B) \to (M, N, ..., R)$+something else
Cross-section is a quadratic expression in terms of scattering amplitudes (matrix elements of scattering matrix)
Inclusive cross-section is a linear expression in terms of matrix elements of inclusive scattering matrix.
Scattering matrix and inclusive scattering matrix contain the same information.

Scattering matrix exists only in theories having particle interpretation. The existence of scattering matrix was proved only in non-relativistic quantum mechanics.
The existence of inclusive scattering matrix can be derived from asymptotic commutativity or cluster property.

Scattering matrix can be obtained from adiabatic scattering matrix.
Inclusive scattering matrix can be obtained from adiabatic scattering matrix in the formalism of L-functionals
Only inclusive scattering matrix makes sense in quantum electrodynamics. L-functionals.

Classical theory with a restricted set of observables

Phase space M, pure states are points of M;
mixed states are probability distributions on M;
every mixed state can be uniquely representedas
a mixture of pure states. Physical observables are
real functions on M. An observable a specifies a
vector field A on M as a Hamiltonian vector field
with the Hamiltonian a:
$Af = \{a, f\}$.
By integrating this vector field we obtain a
one-parameter group $\sigma_A(t)$ of canonical
transformations

Let us suppose that our devices are able to see only a part of observables. We assume that the set Λ of "observable observables" (of real functions on M that can be measured by our devices) is a linear space closed with respect to the Poisson bracket
We label this set by elements of Lie algebra denoted $\mathfrak{g}$. (The map $\gamma \to a_\gamma$ sending $\gamma \in \mathfrak{g}$ into $a_\gamma \in \Lambda$ is an isomorphism of Lie algebras $\mathfrak{g}$ and Λ.) Hamiltonian vector fields A_γ with Hamiltonians a_γ specify an action of Lie algebra $\mathfrak{g}$ on M. The assumption that vector fields A_γ generate one-dimensional subgroups means that this action comes from an action of simply connected Lie group G having $\mathfrak{g}$ as Lie algebra.

One defines the moment map μ of M to $\mathfrak{g}^\vee$ as a map $x \to \mu_x$ where $\mu_x(\gamma) = a_\gamma(x)$. (Here $x \in M, \gamma \in \mathfrak{g}$, and $\mathfrak{g}^\vee$ denotes the space of linear functionals on $\mathfrak{g}$.) This map is G-equivariant with respect to coadjoint action of G on $\mathfrak{g}^\vee$. For every state of the classical system (for every probability distribution ρ on M) we define a point $\nu(\rho) \in \mathfrak{g}^\vee$ as an integral of μ_x over $x \in M$ with respect to the measure ρ:

$$\nu(\rho) = \int_M \mu_x d\rho.$$

The point $\nu(\rho)$ belongs to the convex envelope N of $\mu(M) \subset \mathfrak{g}^\vee$. (The convex envelope of a subset E of topological vector space is defined as the smallest convex closed set containing E.)

We say that two classical states (two probability distributions ρ and ρ') are equivalent if

$$\int_{x\in M} a_\gamma(x)d\rho = \int_{x\in M} a_\gamma(x)d\rho'$$

for every $\gamma \in \mathfrak{g}$. In other words, we say that two states are equivalent if calculations with these states give the same results for every Hamiltonian a_γ. (Our devices cannot distinguish these two states.)
Two states ρ and ρ' are equivalent iff
$\nu(\rho) = \nu(\rho')$.

To give proof we notice that for every $\gamma \in \mathfrak{g}$

$$\nu(\rho)(\gamma) = \int_{x \in M} \mu_x(\gamma) d\rho = \int_{x \in M} a_\gamma(x) d\rho$$

and similarly

$$\nu(\rho')(\gamma) = \int_{x \in M} \mu_x(\gamma) d\rho' = \int_{x \in M} a_\gamma(x) d\rho'.$$

In the classical theory with Hamiltonians taken only from the set $\Lambda = \{a_\gamma\}$ where $\gamma \in \mathfrak{g}$ equivalent states should be identified. The map ν induces a bijective map of the space of equivalence classes onto the set N obtained as a convex envelope of $\mu(M)$ ("quantum states"). The evolution of classical states agrees with the evolution of quantum states.

Complex projective space $\mathbb{CP}$.
Sphere $||x|| = 1$ in complex Hilbert space $\mathcal{H}$ with identifications $x \sim \lambda x$ where $\lambda \in \mathbb{C}, |\lambda| = 1$.
The group U of unitary operators acts transitively on $\mathbb{CP}$. There exists a unique (up to a constant factor) U-invariant symplectic structure on this space.
The manifold $\mathbb{CP}$ can be considered as a coadjoint orbit of the group U (as an orbit of U in the the space dual to the Lie algebra of U).
The symplectic structure on this manifold is the standard symplectic structure on the coadjoint orbit.

"Observable observables" $a_C(x) = \langle x, Cx \rangle$ where C is a self-adjoint operator.
$\mathfrak{g}$ = self-adjoint operators on $\mathcal{H}$ where the operation is defined as the commutator multiplied by i.
The one-parameter group of unitary operators corresponding to the Hamiltonian a_C is given by the formula $\sigma_(t) = e^{-iCt}$.

The moment map transforms a point x into a linear functional on the space of self-adjoint operators that maps an operator C into $\langle x, Cx\rangle$. The convex envelope of the image of the moment map consists of linear functionals of the frorm $\mathrm{Tr}KC$ where K is positive definite self-adjoint operators with unit trace (i.e. it is a density matrix).
We see that by applying our general construction to complex projective space we obtain the conventional quantum mechanics. In this case, our considerations are close to Weinberg's "non-linear quantum mechanics". (Weinberg suggested considering the classical theory on $\mathbb{CP}$ as a deformation of quantum mechanics.)

One can consider a more general case when the manifold M is a coadjoint orbit (an orbit of the group G in the space $\mathfrak{g}^\vee$). It is well known that such an orbit is a homogeneous symplectic manifold. Elements of the Lie algebra $\mathfrak{g}$ can be regarded as linear functions on $\mathfrak{g}^\vee$; let us denote by Λ the set of restrictions of these functions to the orbit. We consider classical theory on M assuming that only observables from the set Λ can be measured. Then eliminating redundant states we obtain a theory where the set of states is a convex envelope of the orbit. (In our case the moment map is simply the embedding of the orbit into $\mathfrak{g}^\vee$.)

Geometric approach to physical theories. Deterministic theories.

In the geometric approach, the starting point is the set of states. We assume that one can consider a mixture of states; therefore we consider the set of states as a closed convex subset $\mathcal{N}$ of Banach space (or, more generally topological vector space) denoted by $\mathcal{L}$. (Instead of this set one can consider convex cone $\mathcal{C}$ of not necessarily normalized states where proportional states are identified.) We fix a subgroup $\mathcal{V}$ of the group of automorphisms of $\mathcal{N}$.

An observable in the geometric approach is specified by a pair (A, a). Here a is a linear functional on $\mathcal{L}$ and A is an element of the Lie algebra of the group $\mathcal{V}$. We say that a is a Hamiltonian function and A is a "Hamiltonian". The set of observables is denoted by Λ.
We define the evolution operator $\sigma_A(t)$ as a solution of the equation of motion:

$$\frac{d\sigma_A(t)}{dt} = A\sigma_A(t)$$

We assume that the Hamiltonian function a is invariant with respect to $\sigma_A(t)$ (equivalently $a(Ax) = 0$.)
One can consider also a more general case when A depends on t.

We considered classical theory with a restricted set of observables. It is equivalent to "quantum" system in geometric approach. The set of states $\mathcal{N}$ should be identified with convex envelope N of $\mu(M)$, the group G plays the role of $\mathcal{V}$. The pairs (A_γ, a_γ) where $\gamma \in \mathfrak{g}$ play the role of observables.

In textbook quantum mechanics $\mathcal{N}$ consists of density matrices (positive definite self-adjoint operators with unit trace acting in Hilbert space $\mathcal{H}$) and the equation of motion in the case of time- independent Hamiltonian $\hat{A}$ has the form

$$\frac{dK}{dt} = AK = -i(\hat{A}K - K\hat{A}). \qquad (1)$$

The space $\mathcal{L}$ consists of all self-adjoint operators having trace (belonging to trace class) and the complexification of this space consists of all operators belonging to trace class. The cone $\mathcal{C}$ consists of positive definite trace class operators. The group $\mathcal{V}$ is isomorphic to the group of unitary operators in $\mathcal{H}$; these operators act on $\mathcal{L}$ by the formula $K \to U^{-1}KU$.

Observables (A, a) correspond to self-adjoint operators $\hat{A}$. The Hamiltonian function $a(K)$ is defined as a linear functional on $\mathcal{L}$ given by the formula $a(K) = tr\hat{A}K$.

In the algebraic approach to quantum theory the starting point is a $*$-algebra $\mathcal{A}$ (a unital associative algebra with involution *) .The cone $\mathcal{C}$ consists of positive linear functionals on $\mathcal{A}$. (One says that a linear functional ω is positive if $\omega(x^*x) \geq 0$ for every $x \in \mathcal{A}$). The set $\mathcal{N}$ consists of positive functionals obeying the normalization condition $\omega(1) = 1$. The space $\mathcal{L} = \mathcal{A}^\vee$ consists of all linear functionals on $\mathcal{A}$. The group $\mathcal{V}$ can be interpreted as the group of automorphisms of $\mathcal{A}$ acting naturally on the dual space $\mathcal{L}$.
An observable (A, a) corresponds to a self-adjoint element $\hat{A}$ of $\mathcal{A}$, or , more generally to a self-adjoint derivation A of $\mathcal{A}$. The Hamiltonian function $a(K)$ where $K \in \mathcal{L}$ is defined by the formula $a(K) = K(\hat{A})$.

The evolution operator $\sigma_A(t)$ acts on $\mathcal{A}$ as conjugation with $e^{i\hat{A}t}$; this action induces an action of $\sigma_A(t)$ on $\mathcal{L} = \mathcal{A}^\vee$. Similarly, the "Hamiltonian" A entering the equation of motion acts on $\mathcal{A}$ as a commutator with $\hat{A}$ (up to a factor of i); this action induces an action of A on $\mathcal{L}$.

If $\mathcal{A}$ is a topological algebra one should assume that all functionals and operators are continuous.

If $\mathcal{A}$ is a C^*-algebra then $e^{i\hat{A}t}$ is a well defined unitary element of $\mathcal{A}$, hence the equation of motion has a solution and the evolution operator $\sigma_A(t)$ is well defined.

We say that there exist redundant states in the theory if on can find such states $x, y \in \mathcal{N}$ that for every observable (A, a) we have $a(x) = a(y)$ (there are no observables that allow us to distinguish these states). In this case, it is useful to work with theory without redundant states (to eliminate redundant states). To construct such a theory we introduce an equivalence relation in $\mathcal{L}$ saying that $x \sim y$ if $a(x) = a(y)$ for every observable (A, a). In the new theory the set of states $\mathcal{N}'$ is defined as a set of equivalence classes in $\mathcal{N}$ The group $\mathcal{V}$ acts on $\mathcal{L}'$ (on the space of equivalence classes in $\mathcal{L}$); its elements can be regarded as automorphisms of $\mathcal{N}'$. The observables descend to $\mathcal{N}'$.

Decoherence. Probabilities

Let us start with some considerations in the framework of textbook quantum mechanics. Let us assume that the quantum mechanical system we consider is placed in random environment. We interpret the random environment as random adiabatic perturbation $\hat{H}(t)$ of the Hamiltonian $\hat{H}$. We assume for simplicity that the operators $\hat{H}(t)$ have simple eigenvalues $E_n(t)$ with orthonormal system of eigenvectors $\phi_n(t)$.

The density matrix $K(t)$ obeys the equation $i\hbar\frac{dK(t)}{dt} = \hat{H}(t)K(t) - K(t)\hat{H}(t)$.
We can solve this equation in adiabatic approximation. Denoting by $K_{mn}(t)$ the matrix entries of $K(t)$ in the basis $\phi_n(t)$ we obtain in this approximation
$\frac{dK_{mn}(t)}{dt} = \frac{i(E_m(t)-E_n(t))}{\hbar}K_{mn}(t)$.
We see that the diagonal entries do not depend on time, but the non-diagonal entries acquire a random phase factor: $K_{mn}(t) = e^{iC_{mn}(t)}K_{mn}(0)$ where $\frac{dC_{mn}(t)}{dt} = \frac{E_m(t)-E_n(t)}{\hbar}$.
This means that only diagonal matrix elements of density matrix K are predictable; this effect is known as decoherence.

Imposing some conditions on the random Hamiltonian $H(t)$ we obtain that the expectation values of non-diagonal matrix entries vanish. In other words the interaction with random environment "kills" non-diagonal entries of density matrix (this corresponds to "the collapse of wave function " of Copenhagen interpretation). At the end we obtain a diagonal density matrix that can be interpreted as the mixture of pure states ϕ_n with probabilities K_{nn}. If K corresponds to a pure state we obtain the standard formula for probabilities.

Very similar arguments can be applied in geometric approach if we impose an additional condition that the set of states $\mathcal{N}$ is bounded.
Let us consider an observable (A, a).
It follows from boundedness of $\mathcal{N}$ that eigenvalues of the "Hamiltonian" A are purely imaginary and A does not have Jordan cells of size > 1. Therefore we assume that A is diagonalizable and has discrete spectrum with eigenvectors ψ_j and eigenvalues ϵ_j.

Let us suppose that $A(g)$ is a continuous family of "Hamiltonians" such that $A(0) = A$. Then for a right choice of eigenvectors (ψ_j) and for $|g| < \delta_j$ we can construct vectors $(\psi_j(g))$ that depend continuously on g in such a way that

$$A(g)\psi_j(g) = \epsilon_j(g)\psi_j(g)$$

where $\psi_j(0) = \psi_j$. (We assume that all non-zero eigenvalues are at most finitely degenerate)
We say ψ_j is a robust zero mode of A if $\epsilon_j(g) \equiv 0$. In other words a zero mode ψ_j is a robust zero mode of A if in any neighborhood of ψ_j and sufficiently small g we can find a zero mode of $A(g)$ in this neighborhood.

Let us model the interaction with environment by random adiabatic " Hamiltonian" $A(g(t))$. Then in the adiabatic approximation

$$\sigma(t)\psi_j = e^{\rho_j(t)}\psi_j(g(t)),$$

where $\frac{d\rho_j}{dt} = \epsilon_j(g(t))$. (In adiabatic approximation we can neglect the derivative $\dot{g}(t)$. This means that $\sigma(t)$ obeys the equations of motion in this approximation.)
Imposing some conditions on the random "Hamiltonian" $A(g(t))$ one can prove that in average the random phase factors $e^{\rho_j(t)}$ vanish unless ϕ_j is a robust zero mode.

Let us sketch the proof of this fact if $g(t) = \alpha t, \alpha \to 0$ and the probability distribution on "Hamiltonians" comes from a probability distribution on the parameter g. In this case the expectation value of the random phase factor can be written in the form

$$\int d\mu e^{\frac{1}{\alpha}\rho_j(g)}$$

where $\frac{d\rho_j}{dg} = \epsilon_j(g)$.
Assuming that the measure μ (the probability distribution on g) is absolutely continuous and taking into account that $\rho_j(g)$ is purely imaginary we obtain that the expectation value tends to zero as $\alpha \to 0$.

Let us assume that all zero modes of A are robust. The operator P defined by the formula

$$Px = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \sigma_A(t)x$$

where $\sigma_A(t)$ is the group of automorphisms generated by A sends all eigenvectors of A that are not zero modes to zero.

To calculate probabilities of the observable (A, a) in the state x we should represent the zero mode Px as a mixture of pure zero modes:

$$Px = \sum p_k z_k.$$

Then p_k is the probability to find the the value $a(z_k)$ measuring the observable (A, a). (We assume that the numbers $a(z_k)$ are different. If this condition is not satisfied we should calculate the probability to obtain the value α summing all p_k with $a(z_k) = \alpha$.)

In the textbook quantum mechanics, we take A as a commutator with a self-adjoint operator $\hat{A}$ multiplied by i and define $a(K) = tr\hat{A}K$. Notice that $a(K)$ is not necessarily finite; for example, in a translation-invariant state the value of energy is in general infinite (but we can talk about the density of energy and about the difference of energies). The situation in the geometric approach is similar.

In $\hat{A}$-representation the basis of eigenvectors of A consists of matrices having only one non-zero entry equal to 1. Diagonal matrices are zero modes of A.

Notice, the representation of zero mode as a mixture of pure zero modes, in general, is not unique. However, in conventional quantum mechanics, this representation is unique in the case when A corresponds to an operator $\hat{A}$ having simple eigenvalues (in this case all zero modes of A are diagonal matrices, all of them are robust).

Suppose that the symplectic manifold M is a coadjoint orbit of Lie group G and the set of "observable observables" is identified with the Lie group $\mathfrak{g}$ of G. (To every element $\gamma \in \mathfrak{g}$ we assign a pair (A_γ, a_γ) where a_γ is the restriction to the orbit of linear functional on $\mathfrak{g}^\vee$ specified by γ and the "Hamiltonian" A_γ comes ffrom coadjoint representation of $\mathfrak{g}$.) Then the space of states of the theory obtained by the elimination of redundant states is a convex envelope N of the orbit; pure states belong to the orbit.

For compact Lie group the coadjoint representation can be identified with adjoint representation. Without loss of generality, we can assume that the "Hamiltonian" A_γ corresponds to an element γ belonging to Cartan subalgebra $\mathfrak{h}$. Using commutativity of Cartan subalgebra we obtain that elements of $\mathfrak{h}$ are zero modes of A_γ. Elements of $\mathfrak{h}$ belonging to an orbit are pure zero modes. If γ is a regular element of Cartan subalgebra $\mathfrak{h}$ all zero modes of A_γ can be considered as elements of $\mathfrak{h}$; all of them are robust. The operator P can be interpreted as an orthogonal projection of $\mathfrak{g}$ onto Cartan subalgebra $\mathfrak{h}$.

If G is a unitary group then the elements of its Lie algebra considered as matrices are regular iff all eigenvalues are distinct. Let us consider a zero mode x (= a stationary state) of Hamiltonian" A_γ corresponding to regular γ; it can be identified with an element of the Cartan subalgebra containing γ. Probabilities in the state x can be calculated as coefficients in the representation of Px as a mixture of pure zero modes.

In the case when the orbit can be identified with complex projective space we obtain conventional quantum mechanics. Notice that in this case, the representation of Px as a mixture of pure zero modes is unique (as should be in quantum theory in the case of simple spectrum). For all other orbits the representation is not unique. This means that corresponding physical theories cannot be described in the framework of conventional quantum mechanics.