L-functionals

The main idea of the geometric approach is to take as a starting point the set of states. It is important to emphasize that working with the set of states is as convenient as working with Hilbert spaces (and sometimes more convenient). We illustrate this by representing states by functionals called L-functionals.

Let us quantize a classical theory with finite or infinite number of degrees of freedom. The Hamiltonian of the classical theory is a functional on the phase space with coordinates p_k, q^k having standard Poisson brackets; after quantization, we obtain operators $\hat{p}_k, \hat{q}^k$ obeying canonical commutation relations (CCR):
$[\hat{p}_k, \hat{p}_{k'}] = 0$, $[\hat{q}^k, \hat{q}^{k'}] = 0$, $[\hat{p}_k, \hat{q}^{k'}] = \frac{\hbar}{i}\delta_k^{k'}$.
It will be convenient to work with CCR for operators $\hat{a}(k) = \frac{\hat{p}_k + \hat{q}^k}{\sqrt{2}}$, $\hat{a}^+(k) = \frac{\hat{p}_k - \hat{q}^k}{\sqrt{2}}$ that can be regarded as generalized functions of continuous and discrete parameters.

In other words, we are working with operators $\hat{a}(f) = \int dk f(k)\hat{a}(k), \hat{a}^+(f) = \int dk f(k)\hat{a}^+(k)$ where f runs over the space of test functions $\mathcal{E}$ considered as pre-Hilbert space. The integral over k is considered as an integral over continuous parameters and a sum over discrete parameters. We assume that $\mathcal{E}$ is the space $\mathcal{S}$ of smooth fast-decreasing functions taking values in $\mathbb{C}^r$.

The CCR can be written in the form

$$[\hat{a}(f), \hat{a}(g)] = [\hat{a}^+(f), \hat{a}^+(g)] = 0, [\hat{a}(f), \hat{a}^+(\bar{g})] = \hbar\langle f, g\rangle$$

where $f, g \in \mathcal{E}$.

In the case of an infinite number of degrees of freedom, there exist representations of CCR that are not equivalent to the standard Fock representation where $\hat{a}(f), \hat{a}^{+}(f)$ can be interpreted as annihilation and creation operators (i.e. there exists a cyclic vector θ obeying $\hat{a}(f)\theta = 0$)). In Hilbert space approach vectors and density matrices in all representation spaces can be regarded as states of the theory at hand.

In the geometric approach, we can represent the states as functionals

$$\mathbf{L}_K(f) = Tr\hat{W}_f K.$$

where $\hat{W}_f = e^{-\hat{a}^+(f)}e^{\hat{a}(\bar{f})}$.

It is easy to verify that this functional is well-defined for a density matrix K in any representation of CCR. The operator $-i\hat{a}^{+}(f) + i\hat{a}(\bar{f})$ is self-adjoint; this is a rigorous form of the statement the operator $\hat{a}^{+}(f)$ is Hermitian conjugate to the operator $\hat{a}(\bar{f})$. Using the fact that f is square integrable we obtain that up to a finite factor the operator $\hat{W}_f$ coincides with unitary operator $e^{-\hat{a}^{+}(f)+\hat{a}(\bar{f})}$.

One can say that when working with functionals $\mathbf{L}$ we consider all representations of CCR simultaneously.
To emphasize that $\mathbf{L}_K$ does not depend analytically on f we use the notation $\mathbf{L}_K(\bar{f}, f)$ or $\mathbf{L}_K(f^*, f)$.
The condition $TrK = 1$ leads to normalization condition $\mathbf{L}_K(0, 0) = 1$.
The operator K is positive definite, this implies some positivity conditions on $\mathbf{L}_K(\bar{f}, f)$.
We denote by $\mathcal{L}$ the vector space of non-linear continuous functionals $\mathbf{L}(\bar{f}, f)$ and by $\mathcal{N}$ the subset of $\mathcal{L}$ satisfying the conditions above.

Algebraic approach.
$*$-algebra=unital associative algebra $\mathcal{A}$ with involution *
State=linear functional ω on $\mathcal{A}$ obeying $\omega(x^*x) \geq 0$ for $x \in \mathcal{A}$ (positive linear functional)
$\mathcal{C}$-cone of states, $\mathcal{N}$ -set of all normalized states $(\omega(1) = 1)$

Weyl algebra=algebra generated by $a(f), a^+(f)$ obeying CCR.
Exponential form of Weyl algebra = algebra $\mathcal{W}$ of operators in Fock space containing all operators of the form W_f and closed in norm topology.
We use the geometric approach taking $\mathcal{N}$ as the set of normalized positive linear functionals σ on $\mathcal{W}$ represented by non-linear functionals $\sigma(W_f)$ on $\mathcal{E}$.
The space $\mathcal{L}$ should be identified with the space of linear functionals on $\mathcal{W}$ or with the space of non-linear functionals on $\mathcal{E}$.

It is easy to check that
$\tilde{b}^+(f)\mathbf{L}_K = \mathbf{L}_{Ka(f)}, \quad \tilde{b}(f)\mathbf{L}_K = \mathbf{L}_{Ka^+(f)},$
$b(f)\mathbf{L}_K = \mathbf{L}_{a(f)K}, \quad b^+(f)\mathbf{L}_K = \mathbf{L}_{a^+(f)K}$
where

$$b(f) = -\hbar c_2^+(f) + c_1(f), \quad b^+(f) = -c_2(f),$$

$$\tilde{b}^+(f) = \hbar c_1^+(f) - c_2(f), \quad \tilde{b}(f) = c_1(f),$$

$c_1^+(\bar{f})$ is a multiplication operator by $\bar{f}$, $c_2^+(f)$ is a multiplication operator by f, and $c_1(\bar{f}), c_2(f)$ are variational derivatives with respect to $\bar{f}$ and f.

More generally, in algebraic approach we start with $*$-algebra $\mathcal{A}$, $\mathcal{L}$ is the space dual to $\mathcal{A}$,every element $\hat{A} \in \mathcal{A}$ specifies two operators A and $\tilde{A}$ acting in $\mathcal{L}$. The operator A transforms $\omega \in \mathcal{L}$into functional $\omega(xA)$, the operator $\tilde{A}$ transforms it into $\omega(A * x)$ It is easy to verify that the sets $\mathcal{N}$ and $\mathcal{C}$ are invariant under the operators of the form $e^{tA}e^{t\tilde{A}}$. In other words, if the equation of motion in the form

$$\frac{d\sigma}{dt} = (A + \tilde{A})\sigma,$$

then $\mathcal{N}$ and $\mathcal{C}$ are invariant under the evolution operator specified by a "Hamiltonian" $A + \tilde{A}$.

Unfortunately, a simple description of the set $\mathcal{N}$ for the algebra $\mathcal{W}$ does not exist. For our goals it is convenient to consider this set as a minimal closed subset of $\mathcal{L}$ invariant under all operators of the form $\exp(A + \tilde{A})$ and containing Gaussian functionals $e^{-\langle f, Sf \rangle}$ where S is a positive definite linear operator

Quantizing classical Hamiltonian we obtain a formal expression

$$\hat{H} = \sum_{m,n} \int \Gamma_{m,n}(k_1, \ldots k_m | l_1, \ldots, l_n) \times$$

$$\hat{a}^+(k_1) \ldots \hat{a}^+(k_m) \hat{a}(l_1) \ldots \hat{a}(l_n) d^m k d^n l.$$

We presented it in the normal form (i.e. all creation operators are moved to the left). In many interesting cases, this formal expression does not define a self-adjoint operator in Fock space, but the corresponding equation of motion in the space $\mathcal{L}$ of functionals $\mathbf{L}(\bar{f}, f)$ is well-defined.

To write down this equation we calculate the operator corresponding in $\mathcal{L}$ to the commutator of $\hat{H}$ with K. We obtain
$\frac{d\mathbf{L}}{dt} = (H + \tilde{H})\mathbf{L}$ where
$H = \frac{1}{i\hbar}(\sum_{m,n} \int \Gamma_{m,n}(k_1, ...k_m|l_1, ...l_n) \times$
$b^+(k_1)...b^+(k_m)b(l_1)...b(l_n)d^m k d^n l)$
$\tilde{H} = \frac{-1}{i\hbar}(\sum_{m,n} \int \bar{\Gamma}_{m,n}(k_1, ...k_m|l_1, ...l_n) \times$
$\tilde{b}^+(k_1)...\tilde{b}^+(k_m)\tilde{b}(l_1)...\tilde{b}(l_n))d^m k d^n l)$

Let us consider in more detail quadratic Hamiltonian

$$\hat{H}(0) = \int \epsilon(k)\hat{a}^{+}(k)\hat{a}(k)dk.$$

Here $k = (\mathbf{k}, s)$ where $\mathbf{k}$ runs over $\mathbb{R}^d$, and s runs over a finite set X, integration over k is understood as integration over $\mathbf{k}$ and summation over discrete intex s. This Hamiltonian commutes with momentum operator $\mathbf{P} = \int \mathbf{k}\hat{a}^{+}(k)\hat{a}(k)dk$, hence it is translation-invariant.
The functionals of the form

$$\mathbf{L}_n = e^{-\int \bar{f}(k)n(kf(k)dk},$$

are translation-invariant stationary states of the corresponding "Hamiltonian"

In particular, equilibrium states of the "Hamiltonian" $H(0)$ have this form. To prove this we make volume cutoff (we replace $\mathbb{R}^d$ with a lattice). Then we can approximate $\hat{H}_0$ by a Hamiltonian of the form $\sum \epsilon_k \hat{a}_k^+ \hat{a}_k$ where k runs over a discrete set. The equilibrium state of the latter Hamiltonian can be represented by density matrix $\Omega(T) = \exp(-\beta \sum \epsilon_k \hat{a}_k^+ \hat{a}_k)/Z$ in the Fock space; it is easy to check that

$$\hat{a}_k \Omega(T) = e^{-\hbar \frac{\epsilon_k}{T}} \Omega(T) \hat{a}_k, \hat{a}_k^+ \Omega(T) = e^{\frac{\hbar \epsilon_k}{T}} \Omega(T) a_k^+.$$

Using these equations we obtain equations for the corresponding L-functional; taking the limit we obtain for the L-functional $\mathbf{L}_T$ corresponding to the equilibrium state in infinite volume

$$c_1(k)\mathbf{L}_T = e^{-\frac{\hbar\epsilon(k)}{T}}(-\hbar c_2^+(k) + c_1(k))\mathbf{L}_T$$

$$c_2(k)\mathbf{L}_T = e^{-\frac{\hbar\epsilon(k)}{T}}(-\hbar c_1^+(k) + c_2(k))\mathbf{L}_T$$

hence

$$n(k) = \frac{\hbar}{e^{\frac{\hbar\epsilon(k)}{T}} - 1}.$$

Functional integrals for evolution operators.
Adiabatic scattering matrix
$\hat{H}(t) = \hat{H}_0 + g(t)\hat{V}$
Adiabatic scattering matrix in the formalism of L-functionals.
Scattering matrix and inclusive scattering matrix.
Perturbation theory.

GGreen functions

$$\langle \alpha | T(C_1(\mathbf{x}_1, t_1) \dots C_s(\mathbf{x}_s, t_s) | \omega \rangle$$

Here $\omega \in \mathcal{L}$, $\alpha \in \mathcal{L}^\vee$ are translation-invariant and stationary, C_i denotes A_i or $\tilde{B}_i$
We assume that spatial and time translations act in $\mathcal{L}$

In algebraic approach we can take $\alpha \in \mathcal{A}$.Then GGreen function can be written as

$$\omega(M\alpha N)$$

where M is chronological product (times decreasing) and N is anti-chronological product (times increasing).
For $\alpha = 1$ we obtain Keldysh Green functions.

Infrared problem in quantum electrodynamics

Let us show that in the formalism of L-functionals infrared divergences do not appear.
Consider a time-dependent Hamiltonian
$\hat{H} = \hat{H}_0 + \hat{V} = \hbar \int dk \epsilon(\mathbf{k}) a^+(\mathbf{k}) \cdot a(\mathbf{k})$
$+\hbar \int \frac{d\mathbf{k}}{\sqrt{2\epsilon(\mathbf{k})}} (j(\mathbf{k}, t) \cdot a^+(\mathbf{k}) + j^*(\mathbf{k}, t) \cdot a(\mathbf{k}))$
where $a(\mathbf{k})$ is a vector potential of electromagnetic field with components $a_\mu(\mathbf{k}), \mu = 0, .., 3$ satisfying the Lorenz gauge condition $k^\mu a_\mu(\mathbf{k}) = 0$. The scalar product of two 4-vectors has the form
$p \cdot k = \mathbf{pk} - p_0 k_0$. We use the notation $\epsilon(\mathbf{k}) = |\mathbf{k}|$.

We suppose that the $j(\mathbf{k}, t)$ is a numerical function (Fourier transform of divergence-free current. For example, we can consider an electron moving in a potential field and interacting with a quantized electromagnetic field (we neglect the action of electromagnetic field on the electron).

The equation of motion for L-functional corresponding to this Hamiltonian has the form , $d\mathbf{L}/dt = H\mathbf{L}$ where $H = H_0 + V$ and

$$H_0 = \hbar \int d\mathbf{k}\epsilon(\mathbf{k})(c_1^+(\mathbf{k}) \cdot c_1(\mathbf{k}) - c_2^+(\mathbf{k}) \cdot c_2(\mathbf{k})),$$

$$V = \hbar \int \frac{d\mathbf{k}}{\sqrt{2\epsilon(\mathbf{k})}}(j(\mathbf{k}, t) \cdot c_1^+(\mathbf{k}) + j^*(\mathbf{k}, t) \cdot c_2^+(\mathbf{k}))$$

The corresponding evolution operator will be denoted by $U(t)$. Working in the interaction picture (= making the change of variables $L_I(t) = e^{iH_0 t/\hbar} L(t)$) we reduce the calculation of $U(t)$ to the calculation of the operator $S(t) = e^{-iH_0 t/\hbar} U(t) e^{iH_0 t/\hbar}$ obeying $i\frac{dS}{dt} = VS$ where$V=$

$$\int \frac{d\mathbf{k}}{\sqrt{2\epsilon(\mathbf{k})}} (e^{i\epsilon(\mathbf{k})t} j(\mathbf{k}, t) \cdot c_1^+(\mathbf{k}) + e^{-i\epsilon(\mathbf{k})t} j^*(\mathbf{k}, t) \cdot c_2^+(\mathbf{k}))$$

A solution of this equation can be found in the form
$e^{\sum_{i=1}^{2}(M_{i+}(t)\cdot c_i^{+}+M_{i-}(t)\cdot c_i)} = \prod_{i=1}^{2} e^{M_i(t)}$
where we use the notation
$M_i(t) = M_{i+}(t)\cdot c_i^{+} + M_{i-}(t)\cdot c_i \equiv$
$\int \frac{d\mathbf{k}}{\sqrt{2\epsilon(\mathbf{k})}}(M_{i+}(\mathbf{k},t)\cdot c_i^{+}(\mathbf{k}) + M_{i-}(\mathbf{k},t)\cdot c_i(\mathbf{k})).$

One can prove that
$\frac{d}{dt}e^{M_i(t)} = \big(\dot{M}_{i+}(t)\cdot c_i^+ + \dot{M}_{i-}(t)\cdot c_i +$
$\frac{1}{2}(\dot{M}_{i+}(t)\dot{M}_{i-}(t) - \dot{M}_{i-}(t)\cdot M_{i+}(t))\big)e^{M_i(t)}$. By comparing this expression with equation of motion we get
$\dot{M}_{1+}(t) = j(\mathbf{k},t)e^{i\epsilon(\mathbf{k})t}$;
$\dot{M}_{2+}(t) = j^*(\mathbf{k},t)e^{-i\epsilon(\mathbf{k})t}$;
$\dot{M}_{i-}(t) = 0$.

Using the definition of operators c_i^+ we can write the expression for the solution of the equation of motion in the formalism of L-functionals in the following form

$$L(\alpha^*, \alpha, t) =$$
$$\exp\Big(\int_{t_0}^{t} d\tau \int \frac{d\mathbf{k}}{\sqrt{2\epsilon(\mathbf{k})}} \big(e^{i\epsilon(\mathbf{k})\tau} j(\mathbf{k}, \tau) \cdot \alpha^*(\mathbf{k}) +$$
$$e^{-i\epsilon(\mathbf{k})\tau} j^*(\mathbf{k}, \tau) \cdot \alpha(\mathbf{k}) \big) \Big) L(\alpha^*, \alpha, t_0)$$

This expression agrees with calculations by Kulish and Faddeev who argued that the asymptotic dynamic of the system of charged particles can be described by the evolution operator

$$U_{as}(t) = e^{iH_0 t} e^{i\Phi(t)} e^{R^I(t)}$$

where $\Phi(t)$ is purely real and independent of electromagnetic field so it drops out of the expression for L-functional and $R^I(t)$ can be written in the form

$$R^I(t) = i \int \frac{d\mathbf{k}}{\sqrt{2\epsilon(\mathbf{k})}} \Big(\int^t e^{i\epsilon(\mathbf{k})\tau} j(\mathbf{k},\tau) d\tau \cdot a^+(\mathbf{k}) + \int^t e^{-i\epsilon(\mathbf{k})\tau} j^*(\mathbf{k},\tau) d\tau \cdot a(\mathbf{k}) \Big).$$

The formula for the solution can be rewritten in the form

$L(\alpha^*, \alpha, t) =$

$\exp\Big(\int d\mathbf{k}\sqrt{2\epsilon(\mathbf{k})}(e^{-i\epsilon(\mathbf{k})t}\mathcal{A}(\mathbf{k}, t)\dot{\alpha}^*(\mathbf{k}) +$

$e^{i\epsilon(\mathbf{k})t}\mathcal{A}^*(\mathbf{k}, t)\cdot\alpha(\mathbf{k})\Big)L(\alpha^*, \alpha, t_0)$

We use the notation

$$\mathcal{A}^\mu(\mathbf{k}, t) = \frac{1}{2\epsilon(\mathbf{k})(2\pi)^{\frac{3}{2}}}\int_{t_0}^{t} d\tau(e^{i\epsilon(\mathbf{k})(\tau-t)}j^\mu(\mathbf{k}, \tau).$$

where $\mathcal{A}^\mu(\mathbf{k}, t)$ is the expectation value of electromagnetic potential.

Indeed, it is easy to check that

$$\langle A_\mu(\mathbf{k},t)\rangle = \frac{e^{i\epsilon(\mathbf{k})t}}{\sqrt{2\epsilon(\mathbf{k})}}\tilde{b}_\mu(\mathbf{k})L(\alpha,\alpha^*,t)|_{\alpha=0} =$$

$$\frac{e^{i\epsilon(\mathbf{k})t}}{\sqrt{2\epsilon(\mathbf{k})}}\frac{\delta}{\delta\alpha^*_\mu}L(\alpha,\alpha^*,t)|_{\alpha=0} = \mathcal{A}_\mu(\mathbf{k},t)$$

Let us now turn to calculating the inclusive cross-section of the photon emission. For this purpose, we calculate the expectation value of the operator

$$\rho(\mathbf{k}) = \sum_{i=\pm} (\varepsilon_i^* \cdot a^+(\mathbf{k}))(\varepsilon_i \cdot a(\mathbf{k})),$$

where ε_i are polarizations of outgoing photons. We get

$$dN(\mathbf{k}) = \langle \rho(\mathbf{k}) \rangle d\mathbf{k} =$$

$$\left(\sum_{i=\pm} \varepsilon_i \frac{\delta}{\delta \alpha(\mathbf{k})} \varepsilon_i^* \frac{\delta}{\delta \alpha^*(\mathbf{k})} L(\alpha, \alpha^*, t)|_{\alpha=0} \right) 2\epsilon(\mathbf{k}) d\mathbf{k}$$

$$= \mathcal{A}(\mathbf{k}, t) \cdot \mathcal{A}^*(\mathbf{k}, t) 2\epsilon(\mathbf{k}) d\mathbf{k}$$

If we are interested in the inclusive cross-section of emission of n photons with momenta $k_1, ..., k_n$ then similar calculations lead to the following formula:

$$dN(\mathbf{k}_1, ..., \mathbf{k}_n) = \langle \rho(\mathbf{k}_1, ..., \mathbf{k}_n) \rangle \prod_{i=1}^{n} d\mathbf{k}_i =$$

$$\prod_{i=1}^{n} \mathcal{A}(\mathbf{k}_i, t) \cdot \mathcal{A}^*(\mathbf{k}_i, t) 2\epsilon(\mathbf{k}_i) d\mathbf{k}_i$$

QED

$$S = S_{mat} + S_{ph} + \int dx j^{\mu}(x) A_{\mu}(x)$$

In formalism of L-functionals we have doubling of fields.
Integrate over doubled photon fields