Particles as elementary excitations

To develop scattering theory we need time translations T_τ and spatial translations $T_{\mathbf{a}}$ acting in the cone of states $\mathcal{C} \subset \mathcal{L}$. These notions allow us to define elementary excitations of translation-invariant stationary state ω.
We define elementary space $\mathfrak{h}$ as a space $\mathcal{S}$ of complex vector-valued smooth fast decreasing functions on $\mathbb{R}^d$ where spatial translatioct as shifts of argument: $(T_{\mathbf{a}}f)(\mathbf{x}) = f(\mathbf{x}+\mathbf{a})$ and time translations T_τ are unitary operators commuting with spatial translations. (A scalar product of two functions taking values in $\mathbb{C}^r$ is specified by the formula $\langle f, g\rangle = \sum_1^r \int d\mathbf{x} f_k(\mathbf{x})\bar{g}_k(\mathbf{x})$.)

In momentum representation spatial translation T_a is defined as multiplication by $e^{i\mathbf{ap}}$ and time translation T_τ can be represented as multiplication by $e^{-iE(\mathbf{p})\tau}$ where $E(\mathbf{p})$ is a Hermitian $r \times r$ matrix.
We say that an elementary space is admissible if the matrix $E(\mathbf{p})$ is positive definite.

An elementary excitation of the state ω is specified by a map σ of elementary space $\mathfrak{h}$ into the cone $\mathcal{C}$ commuting with spatial and temporal translations and bounded linear operators $L(\phi), \phi \in \mathfrak{h}$ acting in the space $\mathcal{L}$ and obeying

$$L(\phi)\omega = \sigma(\phi)$$

Notice that we do not suppose that linear operators L depend linearly on ϕ; it is natural to assume that their dependence on ϕ is quadratic, but this assumption will not be necessary to define the scattering matrix.

The operators $L(\phi)$ should satisfy some additional conditions; not very precisely one can say that $L(\phi)$ and $L(\psi)$ should almost commute when the supports of ϕ and ψ in coordinate representation are far away. This assumption can be made precise in various ways.

If the elementary space is admissible we say that the elementary excitation is a particle.
In relativistic theory the action of the group of translations on the cone $\mathcal{C}$ should be extended to the action of the Poincaré group and the state ω should be Poincaré-invariant. Similarly, the Poincaré group should be represented by unitary transformations of an elementary space $\mathfrak{h}$. A relativistic particle is defined as a map of admissible elementary space equipped with an irreducible representation of Poinaré group into the cone $\mathcal{C}$; this map should commute with actions of Poincaré group in $\mathfrak{h}$ and $\mathcal{C}$.

Denote by U_ϕ an open set containing all points of the form $\nabla \epsilon_i(\mathbf{p})$ where $\epsilon_i(\mathbf{p})$ is an eigenvalue of the matrix $E(\mathbf{p})$ entering the definition of elementary space and $\mathbf{p}$ belongs to the support of $\phi(\mathbf{p})$ in momentum representation (we assume that this support is compact) . Then in coordinate representation the function $T_\tau \phi$ is very small outside the set τU_ϕ for large τ. We say that τU_ϕ is an essential support of $T_\tau \phi$ in coordinate representation.

To prove this we estimate

$(T_\tau \phi)(\mathbf{x}) = \int d\mathbf{p} e^{-i\epsilon(\mathbf{p})\tau + i\mathbf{p}\mathbf{x})} \phi(\mathbf{p})$

for large $|\tau|$. We obtain that this expression is

$\leq \frac{C_n}{1+||\mathbf{x}||^n+|\tau|^n}$

when $\mathbf{x} \notin \tau U_\phi$.

Heuristically this follows from the remark that for $\mathbf{x} \notin \tau U_\phi$ stationary points of $-\epsilon(\mathbf{p})\tau + \mathbf{p}\mathbf{x}$ do not appear in stationary phase method

We say that functions $\phi, \psi \in \mathfrak{h}$ do not overlap if the sets U_ϕ and U_ψ do not overlap. In this case essential supports τU_ϕ and τU_ψ of functions $T_\tau \phi$ and $T_\tau \psi$ in coordinate representation are far away for large $|\tau|$.
It follows that the operators $L(T_\tau \phi)$ and $L(T_\tau \psi)$ almost commute for large $|\tau|$.

Scattering. Inclusive scattering matrix

We would like to consider the scattering of elementary excitations. It seems that one cannot define the conventional scattering matrix in the geometric approach, however, there exists a very natural definition of the inclusive scattering matrix. This definition is based on the consideration of operators $L(f,\tau)$ specified by the formula

$$L(f,\tau)=T_\tau(L(T_{-\tau}f))=T_\tau L(T_{-\tau}f)T_{-\tau}$$

where $L(f)$ denotes the operator entering the definition of elementary excitation.

Using the assumption that ω is a stationary state and the assumption that σ commutes with temporal translations it is easy to check that $L(f,\tau)\omega$ does not depend on τ, hence

$$\dot{L}(f,\tau)\omega = 0$$

To define (inclusive) scattering matrix we fix translation-invariant stationary element α of the dual space $\mathcal{L}^{\vee}$. Then the scattering matrix (more precisely, (α, ω) scattering matrix) is defined as a functional

$S_{n',n}(g'_1, ..., g'_{n'}, g_1, ..., g_n) = \lim_{\tau' \to +\infty, \tau \to -\infty}$
$\langle \alpha | L(g'_1, \tau')...L(g'_{n'}, \tau') L(g_1, \tau)...L(g_n, \tau) | \omega \rangle =$
$\lim_{\tau' \to +\infty} \langle \alpha | L(g'_1, \tau')...L(g'_{n'}, \tau') \Lambda(f_1, \cdots, f_n | -\infty)$

Here

$\Lambda(f_1, \cdots, f_n | -\infty) = \lim_{\tau_i \to -\infty} \Lambda(f_1, \tau_1, ..., f_n, \tau_n)$

where

$\Lambda(f_1, \tau_1, \cdots, f_n, \tau_n) = L(f_1, \tau_1)...L(f_n, \tau_n)\omega.$

We say that $\Lambda(f_1, \cdots, f_n | -\infty)$ is an *in*-state.

We assume that the functions $(g_1, ..., g_n)$ *do not overlap as well as the functions* $(g'_1, ..., g'_{n'})$*.*
It follows that essential supports of functions $T_\tau g_i$ and $T_\tau g_j$ are far away for $\tau \to -\infty$. This implies that operators $L(g_i, \tau)$ and $L(g_j, \tau)$ commute in this limit. We conclude that $S_{n,n'}$ is symmetric with respect to $g_1, ..., g_n$; similarly, we can prove that it is symmetric with respect to $g'_1, .., g'_{n'}$.

To prove the existence of the limit as $\tau \to -\infty$ we assume that

$$||[\dot{L}(g_i, \tau), L(g_j, \tau)]|| < c(\tau)$$

where $c(\tau)$ is a summable function. Then the derivative of the function under the lim sign with respect to τ is also summable; the existence of the limit follows from this fact. To estimate this derivative we use the Leibniz rule. We obtain n summands, and to prove that every summand is summable we transfer the factor with time derivative to the rightmost place.
One should expect that the condition on the commutators is satisfied if functions g_i, g_j do not overlap.

To prove the existence of the limit as $\tau' \to +\infty$ we impose additional condition
$\langle \alpha | \dot{L}(g, \tau) \to 0$ as $\tau \to +\infty$
and use the same reasoning.

To justify the definition of scattering matrix we analyze the notion of *in*-state.
For large negative τ the state

$$T_\tau \Lambda(f_1, \cdots, f_n| -\infty)$$

can be described as a collection of particles with wave functions $T_\tau f_i$.
To prove this fact we use the formulas

$$T_\tau(L(f, \tau')) = T_{\tau+\tau'} L(T_{-\tau'} f) T_{-\tau-\tau'} = L(T_\tau f, \tau+\tau'),$$

$$T_\tau \Lambda(f_1, \cdots, f_n| -\infty) = \Lambda(T_\tau f_1, \cdots, T_\tau f_n| -\infty).$$

If functions $f_1, ..., f_n$ do not overlap we have a collection of distant particles for $\tau \to -\infty$.

This remark allows us to say that the state $T_\tau \Lambda(f_1, \cdots, f_n| -\infty)$ describes a collision of particles with wave functions $(f_1, \cdots, f_n)$.
A number describing this state in the limit $\tau \to +\infty$ can be interpreted as a matrix element of the inclusive scattering matrix.

Algebraic approach

In the algebraic approach, the starting point is a $*$-algebra $\mathcal{A}$; time shifts and spatial shifts come from automorphisms of $\mathcal{A}$. We can construct the data of the geometric approach identifying the cone $\mathcal{C}$ of states with the cone of positive functionals on $\mathcal{A}$ (of linear functionals obeying $\omega(A^*A) \geq 0$) and the space $\mathcal{L}$ with the space of continuous linear functionals on $\mathcal{A}$. Fixing a translation-invariant stationary element $\alpha \in \mathcal{A} \subset \mathcal{L}^\vee$ and translation-invariant stationary state $\omega \in \mathcal{C}$ we can construct (inclusive) scattering matrix. (For example, we can take $\alpha = 1$.)

However, in algebraic approach we can construct also the conventional scattering matrix starting with a linear map $\Phi : \mathfrak{h} \to \mathcal{H}$ commuting with spatial and temporal translations. (Here $\mathcal{H}$ stands for pre Hilbert space obtained from translation-invariant stationary state ω by means of GNS construction.)
We assume that $\Phi(f) = \hat{B}(f)\theta$ where θ denotes cyclic vector corresponding to ω and $B(f) \in \mathcal{A}$

Define $B(f,\tau) = T_\tau B(T_{-\tau} f) T_{-\tau}$
and *in*-vector
$\Psi(f_1, \cdots, f_n | -\infty) =$
$\lim_{\tau_1 \to -\infty, \cdots, \tau_n \to -\infty} \Psi(f_1, \tau_1, \cdots, f_n, \tau_n)$
where
$\Psi(f_1, \tau_1, ..., f_n, \tau_n) = \hat{B}(f_1, \tau_1) ... \hat{B}(f_n, \tau_n)\theta$
The limit exists in $\bar{\mathcal{H}}$ if functions $f_1, ..., f_n$ do not overlap and the algebra $\mathcal{A}$ is asymptotically commutative:

$$||[\hat{A}(\mathbf{x}, t), \hat{B}]|| \leq \frac{C_n(t)}{1 + ||\mathbf{x}||^n}$$

Møller matrix

$$S(0, -\infty) : \mathcal{H}_{as} \to \bar{\mathcal{H}}$$

$\mathcal{H}_{as}$-Fock space (direct sum of symmetric powers of $\mathfrak{h}$)

$$S(0, -\infty)(f_1 \otimes ... \otimes f_n) = \Psi(f_1, ..., f_n| - \infty)$$

Scattering matrix

$$S = S(0, +\infty)^{-1} S(0, -\infty)$$

makes sense if Møller matrices are surjective. Then we say that the theory has particle interpretation.
out-operators

$$a^*_{out}(f) = \lim_{\tau \to +\infty} \hat{B}(f, \tau), a_{out}(f) = \lim_{\tau \to +\infty} \hat{B}^*(f, \tau),$$

in-operators, $\tau \to -\infty$

Green functions in the state ω=expectation values of chronological products (times decreasing)

$\omega\big(T(A_1(\mathbf{x}_1, t_1)...A_n(\mathbf{x}_n, t_n))\big)$

LSZ

Scattering matrix can be expressed in terms of asymptotic behavior of Green function in $(\mathbf{p}, t)$-representation as $t \to \pm\infty$ or in terms of poles in $(\mathbf{p}, \epsilon)$-representation (Green function on shell)

Relation to geometric approach

$\Phi(f) = \hat{B}(f)\theta$

$(\sigma(f))(A) = \langle \Phi(f), \hat{A}\Phi(f) \rangle =$

$\langle \theta, \hat{B}^*(f)\hat{A}\hat{B}(f)\theta \rangle =$

$\omega(B^*(f)AB(f))$

hence

$L(f) = \tilde{B}(f)B(f)$

$L(f, \tau) = \tilde{B}(f, \tau)B(f, \tau)$

We use notations

$(\tilde{B}\sigma)(A) = \sigma(B^*A), (B\sigma)(A) = \sigma(AB)$

We see that in algebraic approach operators $\lim_{\tau \to +\infty} L(g, \tau)$ can be expressed in terms of *out*-operators $a^*_{out}(f), a_{out}(f)$. Using this remark one can express inclusive cross-sections in terms of the inclusive scattering matrix where α is the unit element of the algebra $\mathcal{A}$

In the algebraic approach inclusive scattering matrix with $\alpha = 1$ can be expressed in terms of generalized Green functions on shell.
Generalized Green functions can be defined by the formula $\omega(NM)$ where in one of the factors we have the chronological product of operators (times decreasing) and in another factor, we have the antichronological product (times increasing).
They appear naturally in Keldysh formalism of non-equilibrium statistical physics and in the formalism of L-functionals

GGreen functions can be defined by the formula $\omega(M\alpha N)$; they can be used to express the (α, ω) scattering matrix.