Asymmetric Simple Exclusion Process: Integrable Structure & Limit Theorems

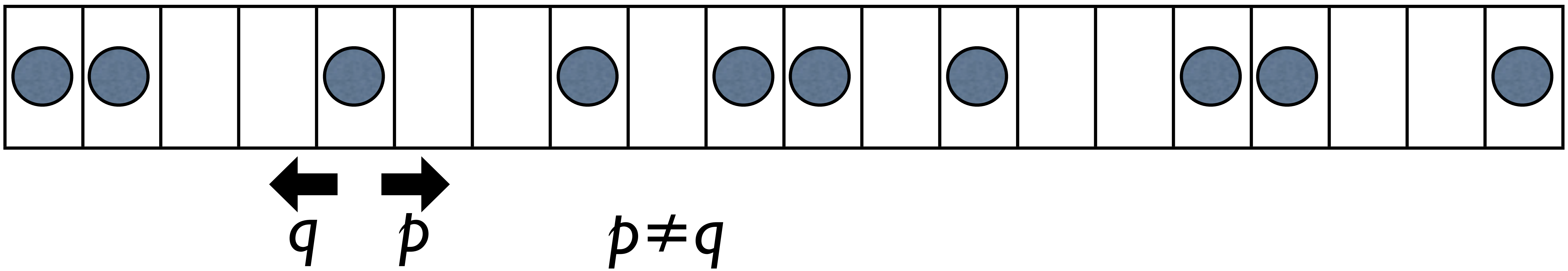
Northeast Probability Seminar

November 2009

Craig Tracy & Harold Widom

ASEP on Integer Lattice

Introduced by F. Spitzer (1970)



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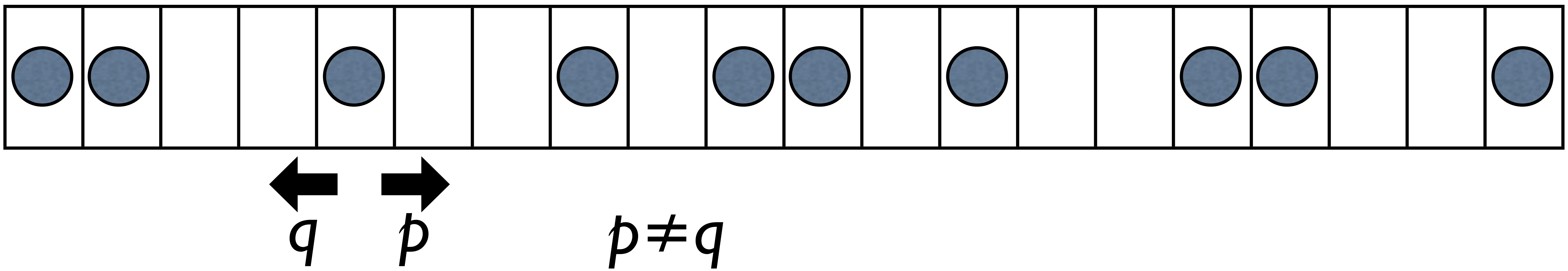

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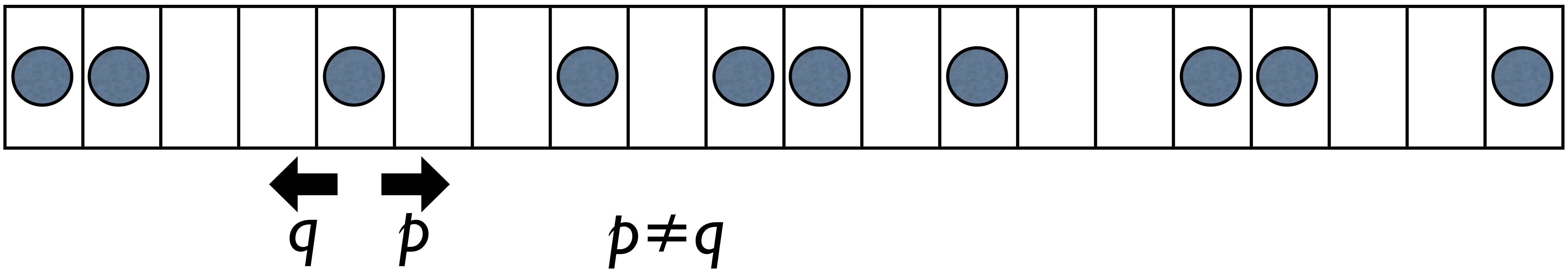

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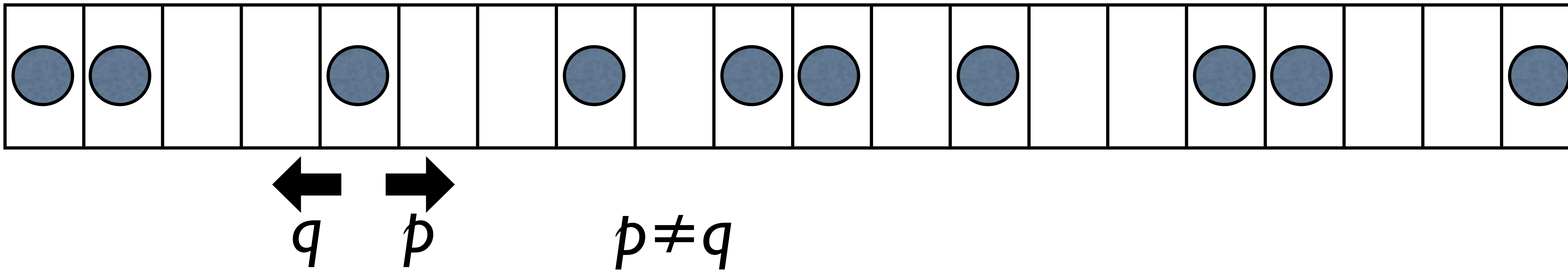

- Each particle has an alarm clock -- exponential distribution with parameter one
- When alarm rings particle jumps to right with probability p and to the left with probability q
- Jumps are suppressed if neighbor is occupied

Initial Conditions

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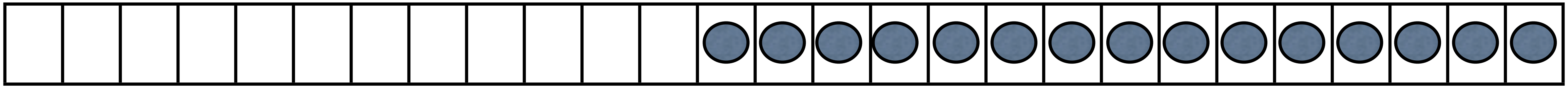

Step Initial Condition, q>p

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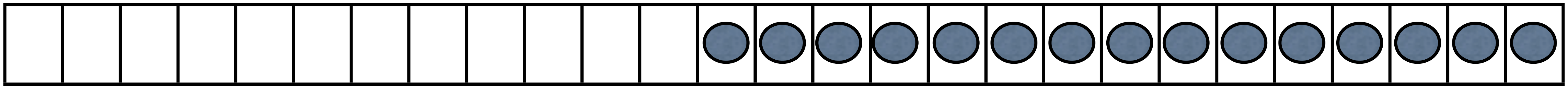

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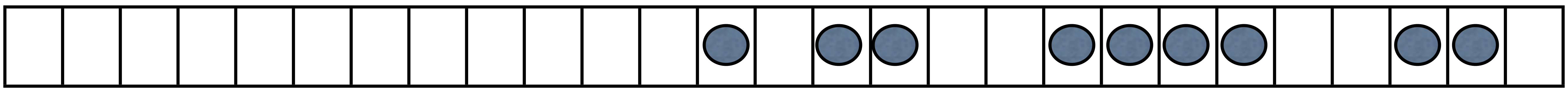

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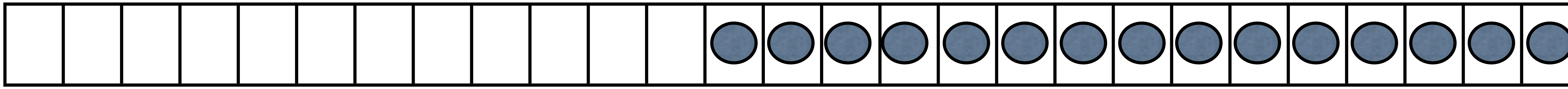

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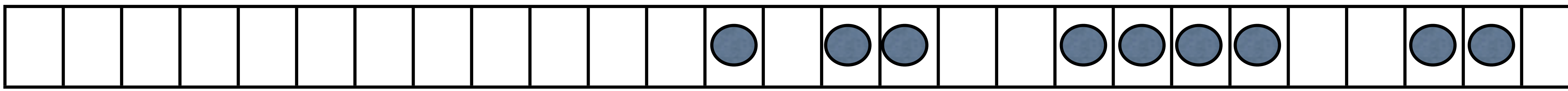

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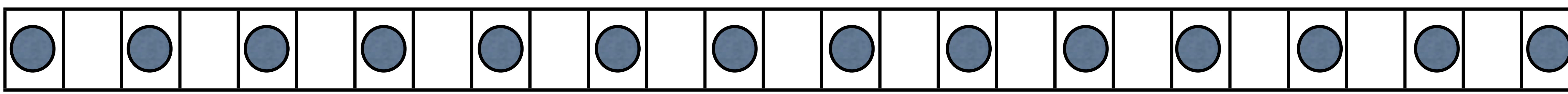

Flat Initial Condition

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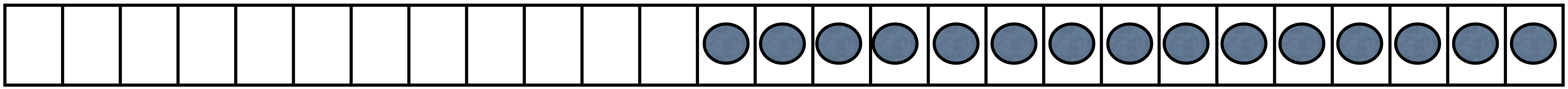

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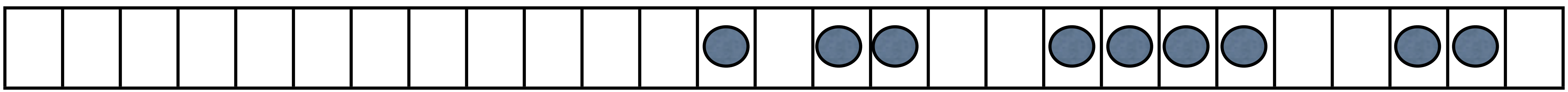

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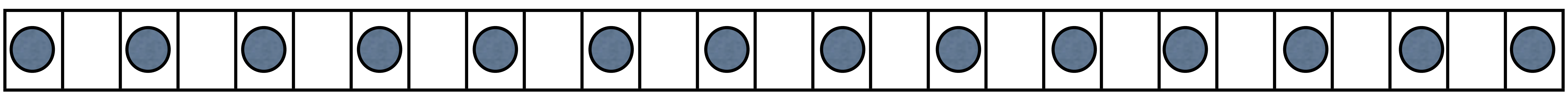

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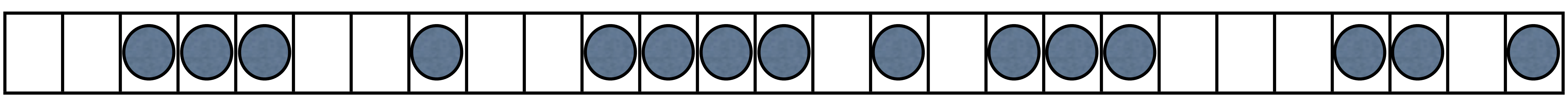

Bernoulli Initial Condition

Current Fluctuations

- Let $I(x,t)$= # of particles less than or equal to x at time t. (Definition OK for step type initial conditions.)
- $I(x,t)$ is a random variable: What can be said about its long-time behavior? Can we find a central limit type theorem for $I(x,t)$?
- First some results for the special case of TASEP where particles are allowed to jump in only one direction

T(Totally)ASEP

- Determinantal Process: Correlations are expressed as determinants of kernel functions K(x,y).
- As a result many techniques from random matrix theory can be applied to TASEP.
- This determinantal structure goes back at least to G. Schütz (1997).
- First limit theorems: K. Johansson (2000), "Shape Fluctuations and Random Matrices" where TASEP was a limiting case of a certain Corner Growth Model.

Current & Position of m^{th} particle

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Step initial condition

$x_m(t)$=position of m^{th} particle from left at time t, $x_m(0)=m$

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Event:

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From this and exclusion property:

$$\text{Prob}(I(x,t)\leq m) = 1-\text{Prob}(x_{m+1}(t)\leq x)$$

TASEP: Step Initial Condition

Theorem (K. Johansson):

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{I([-vt],t) - a_1 t}{a_2 t^{1/3}} \le s\right) = 1 - F_2(-s)$$

where

$$a_1 = \frac{1}{4}(1-v)^2, \;\; a_2 = 2^{-4/3}(1-v^2)^{2/3}, \;\; 0 \le v < 1$$

and F_2 is a distribution function first arising in random matrix theory. It can be expressed in terms of a Painlevé II function or as a Fredholm determinant of the "Airy kernel".

Remarks

- Fluctuations are of order $t^{1/3}$ not $t^{1/2}$.
- This ⅓ exponent is related to the physicists' KPZ Universality (Kardar-Parisi-Zhang)
- The coefficient a_1 goes back to H. Rost (1981).
- Proof relies heavily on the RSK algorithm and the closely associated determinantal structure: "Discrete random martrix ensembles".

TASEP: Step Bernoulli, density ρ<1

Conjectured by M. Prähofer & H. Spohn (2002) & proved by G. Ben Arous & I. Corwin (2009):

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{I([vt],t)-a_1 t}{a_2 t^{1/3}} \leq s\right) = \begin{cases} 1-F_2(-s), & -1<v<2\rho-1, \\ 1-F_1(-s)^2, & v=2\rho-1. \end{cases}$$

where coefficients a_i as before ($v \to -v$) & F_1 is another RMT distribution expressible in terms of the same Painlevé II function.

For $v>2\rho-1$ fluctuations are $t^{1/2}$ and Gaussian.

The transition: $F_2 \longrightarrow F_1^2 \longrightarrow G$

has appeared in other determinantal processes.

The Big Question

Do these results extend to ASEP

and more generally to a wider class of 1D exclusion processes ?

- With regards to the $t^{1/3}$: M. Balázs & T. Seppäläinen have proved this for Bernoulli (stationary) initial conditions for nearest-neighbor ASEP. Quastel & Valkó extend the result to finite-range asymmetric exclusion. Methods use coupling (BS) and comparison estimates (QV). Methods so far do not extend to limit theorems.

- H. Widom & CT show for (nearest neighbor) ASEP the two previous limit theorems remain exactly the same when time t is replaced by $t/(q-p)$ thus proving KPZ Universality. Our methods start with ideas coming from Bethe Ansatz.

Main Steps in Proof of Limit Theorems for ASEP

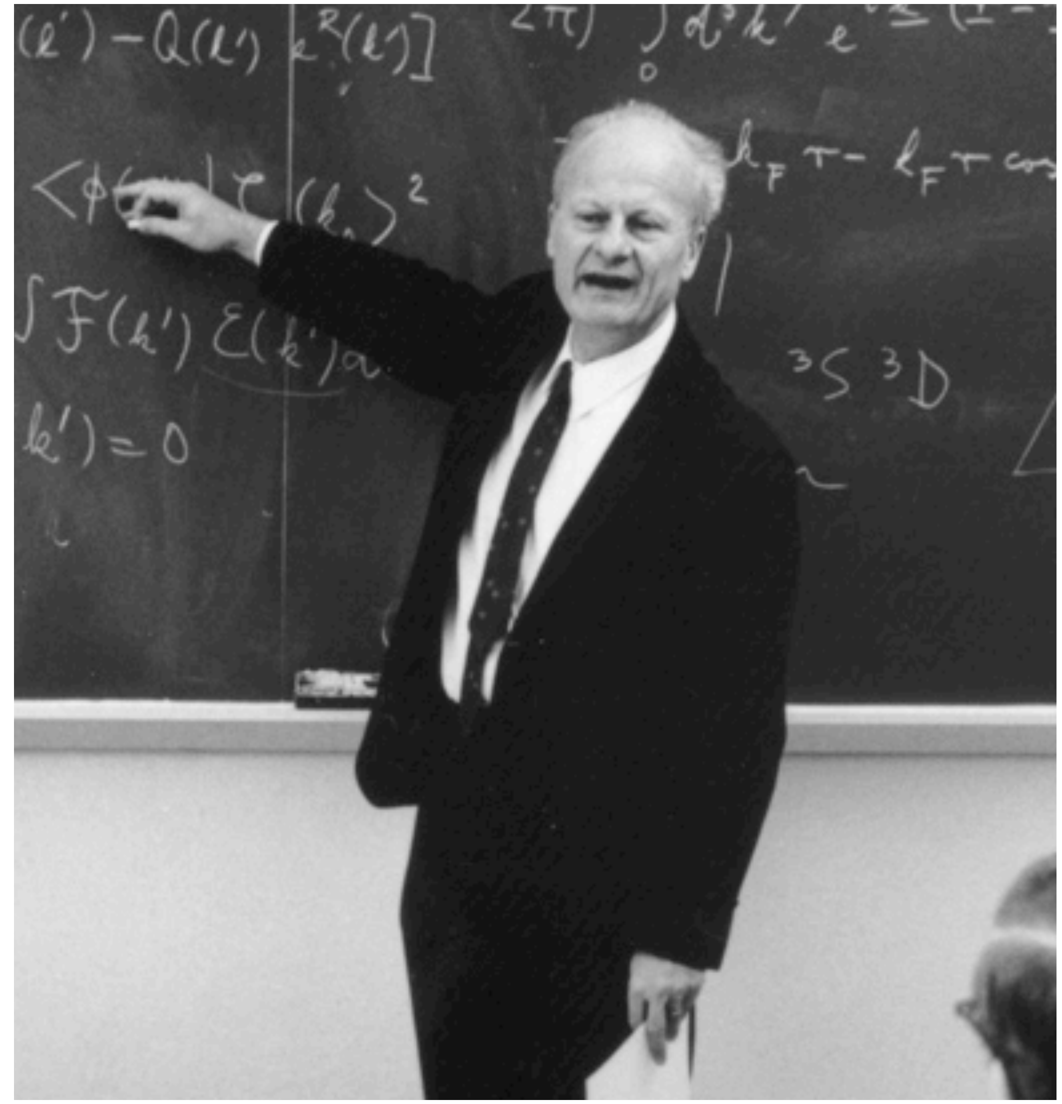
- Use ideas from Bethe Ansatz to find exact expression for the transition probability $Y \to X$ for finite N ASEP.
- Use some amazing combinatorial identities to compute marginal distributions $P(x_m(t) \le x)$. It is then possible to take $N \to \infty$. Marginal expressed as an infinite sum where k^{th} term is a k-dimensional integral.
- Again certain identities permit series to be summed to one contour integral involving a Fredholm determinant.
- Asymptotic analysis of this Fredholm determinant. Kernel initially not of familiar RMT structure.

Integrable Structure of ASEP

We solve the Kolmogorov forward equation ("master equation") for the transition probability Y→X for finite N ASEP:

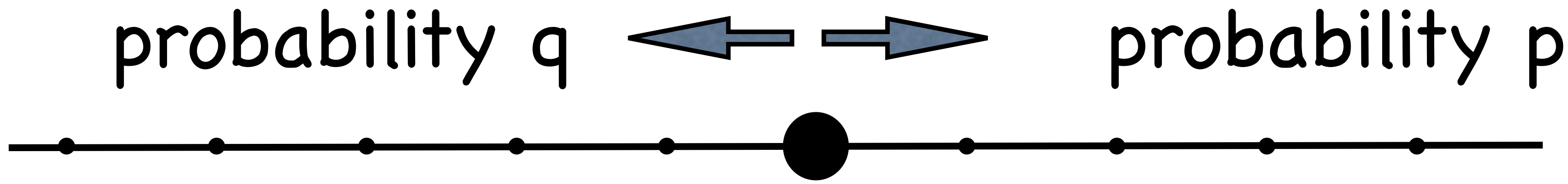
$$P_Y(X;t)$$

Main idea comes from the Bethe Ansatz (1931)



Hans Bethe in 1967

N=1 ASEP



Let $P_Y(X;t)$=probability Y→X at time t.

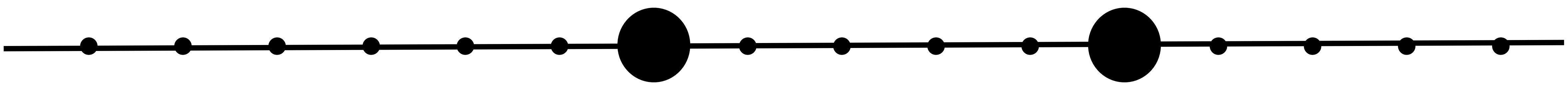
Master equation:

$$\frac{dP}{dt} = p\,P(x-1;t) + q\,P(x+1;t) - P(x;t)$$

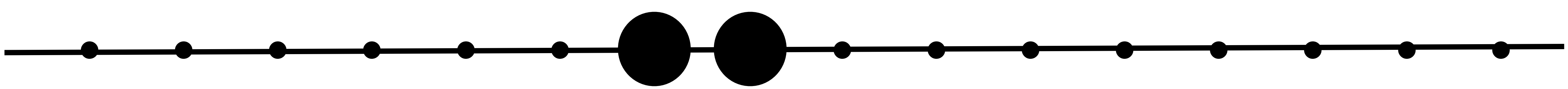
$$P_y(x;t) = \int_{\mathcal{C}} \xi^{x-y-1} e^{t\varepsilon(\xi)}\,d\xi$$

$$\varepsilon(\xi) = \frac{p}{\xi} + q\,\xi - 1$$

N=2 ASEP



Master equation takes simple form for this configuration



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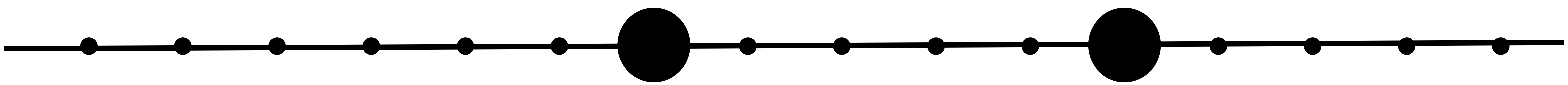

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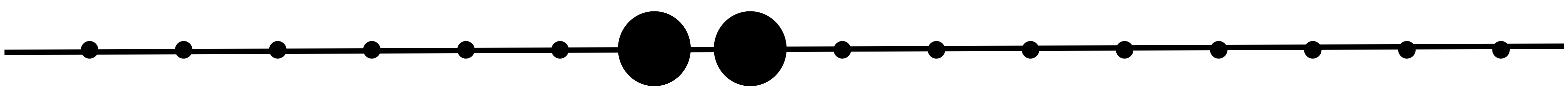


Master equation reflects exclusion for this configuration

→ Impose boundary conditions for first equation so that if satisfied the second equation is automatically satisfied --- Bethe's Idea

N=2 Equations for $P(x_1,x_2;t)$

$x_2 > x_1 + 1$: Not neighbors

$$\frac{dP(x_1,x_2;t)}{dt} = pP(x_1-1,x_2;t) + qP(x_1+1,x_2;t) + pP(x_1,x_2-1;t) + qP(x_1,x_2+1;t) - 2P(x_1,x_2;t)$$

$x_2=x_1+1$: Neighbors

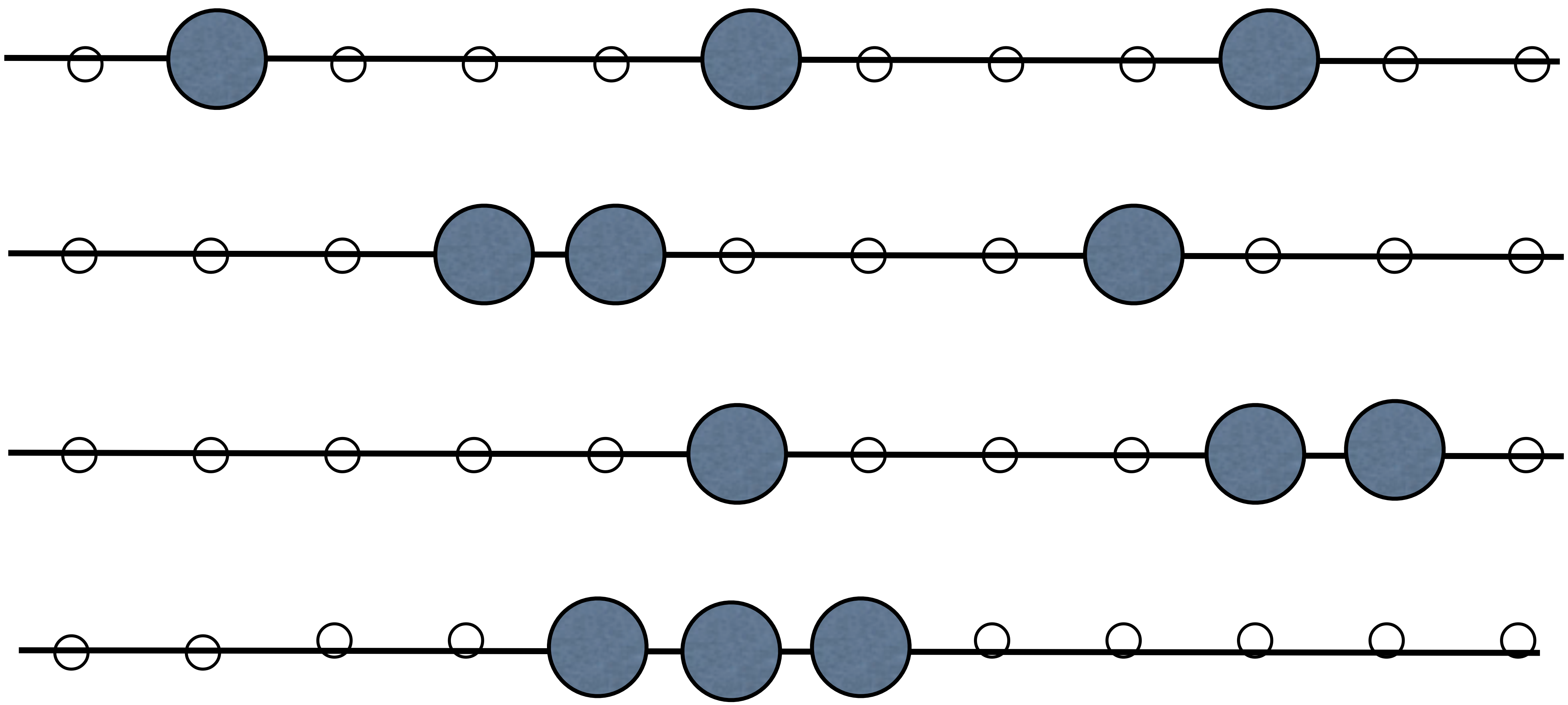
$$\frac{dP(x_1,x_2;t)}{dt} = pP(x_1-1,x_2;t) + qP(x_1,x_2+1;t) - P(x_1,x_2;t)$$

Subtract:

$$qP(x_1+1,x_1+1;t) + pP(x_1,x_1;t) - P(x_1,x_1+1;t) = 0$$

Important Point

New boundary conditions arise for N=3, 4,...



Last configuration requires new BC -- automatically satisfied by 2-particle BC

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For any $\xi_1,\ldots,\xi_N \in \mathbf{C}\setminus\{0\}$ and any permutation σ a solution is

$$\prod_j \xi_{\sigma(j)}^{x_j}\, e^{t\varepsilon(\xi_j)}$$

Can take linear combination or integral of a linear combination & have a solution:

$$\int \sum_{\sigma\in\mathcal{S}_N} F_\sigma(\xi) \prod_j \xi_{\sigma(j)}^{x_j} \prod_j e^{t\varepsilon(\xi_j)}\, d^N\xi$$

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$$\text{If } A_\sigma = \text{sgn}(\sigma) \frac{\prod_{i<j}(p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)})}{\prod_{i<j}(p + q\xi_i\xi_j - xi_i)}$$

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then solution to DE that satisfies BC is

$$\sum_\sigma \int A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{t\varepsilon(\xi_i)}\, d^N\xi$$

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- Term for $\sigma=\mathrm{id}$ satisfies initial condition. Must show remaining $N!-1$ terms give zero at $t=0$. This is the new part of the problem.

- Have not yet specified the contours.

Theorem (TW): If $p \neq 0$ and r is small enough then

$$\mathbb{P}_Y(X;t) = \sum_{\sigma \in \mathcal{S}_N} \int_{\mathcal{C}_r^N} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \left(\xi_i^{-y_i - 1} e^{\varepsilon(\xi_i)\, t} \right) d^N \xi.$$

where

$$A_\sigma = \operatorname{sgn} \sigma \, \frac{\prod_{i<j} (p + q \xi_{\sigma(i)} \xi_{\sigma(j)} - \xi_{\sigma(i)})}{\prod_{i<j} (p + q \xi_i \xi_j - \xi_i)}$$

and satisfies

$$\mathbb{P}_Y(X;0) = \delta_Y(X).$$

Remarks:

- There is no Ansatz in our work!
- Usual Bethe Ansatz calculates the spectrum of the operator. This leads to transcendental equations for the eigenvalues and issues of completeness of the eigenfunctions.
- We compute the semigroup directly. No spectral theory.

Marginal Distributions

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Case m=1:

Fix $x_1=x$, sum $P_Y(X;t)$ over allowed x_2, x_3, x_4,...

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Case m=1:

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Can do this since contours are small: $|\xi_i| < 1$

Result is an expression involving N! terms. Use first miraculous identity to reduce sum to one term!

Here's the identity:

First Identity

$$\sum_{\sigma\in\mathcal{S}_N} \operatorname{sgn}\sigma \left(\prod_{i<j}(p+q\xi_{\sigma(i)}\xi_{\sigma(j)}-\xi_{\sigma(i)})\right.$$

$$\left.\times \frac{\xi_{\sigma(2)}\xi^2_{\sigma(3)}\cdots\xi^{N-1}_{\sigma(N)}}{(1-\xi_{\sigma(2)}\xi_{\sigma(3)}\cdots\xi_{\sigma(N)})(1-\xi_{\sigma(3)}\cdots\xi_{\sigma(N)})\cdots(1-\xi_{\sigma(N)})}\right)$$

$$= p^{N(N-1)/2}\,\frac{(1-\xi_1\cdots\xi_N)\,\prod_{i<j}(\xi_j-\xi_i)}{\prod_i(1-\xi_i)}$$

- Using this identity we get for m=1 an expression for $P(x_1(t) \leq x)$ as a single N-dimensional integral with a product integrand. This expression is for finite-N ASEP

$$I(x, Y, \xi) \quad = \quad \prod_{i<j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \frac{1 - \xi_1 \cdots \xi_N}{(1-\xi_1)\cdots(1-\xi_N)}$$
$$\prod_i \left(\xi_i^{x-y_i-1} e^{\varepsilon(\xi_i)t}\right)$$

Prob(x_1(t)=x) =

$$p^{N(N-1)/2} \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} I(x, Y, \xi)\, d\xi_1 \cdots d\xi_N$$

$$(p \neq 0)$$

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- Sum of N! integrals reduced to one integral
- Form is not so useful to take N→∞
- We now expand contour outwards -- only residues that contribute come from ξ_i=1.
- Can take N→∞ in resulting expression to obtain

$$\sigma(S) := \sum_{i\in S} i$$

$$\mathbb{P}(x_1(t) = x) = \sum_S \frac{p^{\sigma(S)-|S|}}{q^{\sigma(S)-|S|(|S|+1)/2}} \times \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} I(x, Y_S, \xi)\, d^{|S|}\xi$$

The sum is over all nonempty subsets of $\mathbb{Z}^+$
(↑finite)

When p=0 only one term is nonzero, S={1}.

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- These come from the fact that we must sum over all $x_j > x_m$ and all $x_i < x_m$. Some contours must be small (former) and some must be large (latter) to obtain convergence of geometric series
- This involves finding a new identity

Second Identity

S ranges over subsets of $\{1, 2, \ldots, N\}$

$$\sum_{|S|=m} \prod_{i\in S,\, j\in S^c} \frac{p + q\xi_i\xi_j - \xi_i}{\xi_j - \xi_i} \cdot (1 - \prod_{j\in S^c} \xi_j)$$

$$= q^m \begin{bmatrix} N \\ m \end{bmatrix} (1 - \prod_{j=1}^{N} \xi_j).$$

$$[N] = \frac{p^N - q^N}{p - q}, \qquad [N]! = [N]\,[N-1]\cdots[1],$$

$$\begin{bmatrix} N \\ m \end{bmatrix} = \frac{[N]!}{[m]!\,[N-m]!}, \qquad (q - \text{binomial coefficient}),$$

Final series result for case $Y = \mathbb{Z}^+$

$$\mathbb{P}\left(x_m(t) \le x\right) = (-1)^m \sum_{k \ge m} \frac{1}{k!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix}_\tau p^{(k-m)(k-m+1)/2} q^{km+(k-m)(k+m-1)/2}$$

$$\times \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} \prod_{i \ne j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \prod_i \frac{1}{(1-\xi_i)(q\xi_i - p)}$$

$$\times \prod_i \left(\xi_i^x e^{\varepsilon(\xi_i)t} \right) d\xi_1 \cdots d\xi_k$$

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- For p=0 only k=m term is nonzero
- Recognize double product as a determinant whose entries are a kernel, i.e. $K(\xi_i,\xi_j)$

Final series result for case $Y = \mathbb{Z}^+$

$$\begin{aligned}
\mathbb{P}\left(x_m(t) \leq x\right) &= (-1)^m \sum_{k \geq m} \frac{1}{k!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix}_\tau p^{(k-m)(k-m+1)/2} q^{km+(k-m)(k+m-1)/2} \\
&\times \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} \prod_{i \neq j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \prod_i \frac{1}{(1-\xi_i)(q\xi_i - p)} \\
&\times \prod_i \left(\xi_i^x e^{\varepsilon(\xi_i)t} \right) d\xi_1 \cdots d\xi_k
\end{aligned}$$

- For p=0 only k=m term is nonzero
- Recognize double product as a determinant whose entries are a kernel, i.e. $K(\xi_i,\xi_j)$
- Result can then be expressed as a contour integral whose integrand is a Fredholm determinant.

Fredholm determinant

- Let K(x,y) be a kernel function
- Fredholm expansion of det(I-λK):

$$\frac{(-1)^n}{n!}\int\cdots\int \det\left(K(\xi_i,\xi_j)\right)_{1\le i,j\le n}\, d\xi_1\cdots d\xi_n =$$

$$\int_{\mathcal{C}} \det\left(I-\lambda K\right)\,\frac{d\lambda}{\lambda^{n+1}}$$

- Can then do sum over k (q-Binomial theorem):

Final expression for m[th] particle distribution fn.

Step initial condition

Set $\gamma = q - p > 0$, $\tau = p/q < 1$ and define an integral operator K on the circle $\mathcal{C}_R$, $R \gg 1$

$$K(\xi, \xi') = q \, \frac{\xi^x e^{\varepsilon(\xi) t}}{p + q\xi\xi' - \xi}$$

then

$$\mathbb{P}\left(x_m(t/\gamma) \le x\right) = \int \frac{\det\left(I - \lambda K\right)}{\prod_{k=0}^{m-1}\left(1 - \lambda \tau^k\right)} \, \frac{d\lambda}{\lambda}$$

where the contour encloses all the singularities at $\lambda = 0, \tau^{-k}$, $k = 0, \ldots, m-1$.

Final expression for m^{th} particle distribution function

Step Bernoulli initial condition

The expression for the marginal

$$P(x_m(t) \leq x)$$

is the same once we replace K by the new K:

$$K(\xi,\xi') = q\,\frac{\xi^x e^{\varepsilon(\xi)t}}{p+q\xi\xi'-\xi}\;\frac{\rho(\xi-\tau)}{\xi-1+\rho(1-\tau)}$$

$\Uparrow$

new term

Asymptotic analysis

We now transform the operator K so that we can perform a steepest descent analysis.

Recall that the generic behavior for the coalescence of two saddle points leads to the Airy function Ai(x)

George Airy

$$\xi \longrightarrow \frac{1-\tau\eta}{1-\eta}, \quad \tau = \frac{p}{q} < 1,$$

$$K(\xi,\xi') \longrightarrow K_2(\eta,\eta') = \frac{\varphi(\eta')}{\eta' - \tau\eta}$$

$$\varphi(\eta) = \left(\frac{1-\tau\eta}{1-\eta}\right)^x e^{\left[\frac{1}{1-\eta} - \frac{1}{1-\tau\eta}\right]\mathbf{t}}$$

Introduce: $K_1(\eta,\eta') = \dfrac{\varphi(\tau\eta)}{\eta' - \tau\eta}$

Two Preliminary Results

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Proposition:

Let Γ be any closed curve going around $\eta=1$ once counterclockwise with $\eta=1/\tau$ on the outside. Then the Fredholm determinant of $K(\xi,\xi')$ acting on C_R has the same Fredholm determinant as $K_1(\eta,\eta')-K_2(\eta,\eta')$ acting on Γ.

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Proposition:

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Proposition:

Suppose the contour Γ is star-shaped with respect to $\eta=0$. Then the Fredholm determinant of K_1 acting on Γ is equal to

$$\prod_{k=0}^{\infty}(1-\lambda\tau^k)$$

Denote by R the resolvent kernel of K_1

Factor determinant:

$$\det(I-\lambda K)=\det(I-\lambda K_1)\det(I+K_2(I+R))$$

Set $\lambda=\tau^{-m}\mu$ so formula for distr. fn becomes

$$\int \prod_{k=0}^{\infty}(1-\mu\tau^k)\det\left(I+\tau^{-m}\mu K_2(I+R)\right)\ \frac{d\mu}{\mu}$$

μ runs over a circle of radius $> \tau$

By a perturbative expansion of R, followed by a deformation of operators, we show

$$\begin{aligned}
\det\left(I+\lambda K_2(I+R)\right) &= \det\left(I+\mu J\right) \\
J(\eta,\eta') &= \int \frac{\varphi_\infty(\zeta)}{\varphi_\infty(\eta')}\,\frac{\zeta^m}{(\eta')^{m+1}}\,\frac{f(\mu,\zeta/\eta')}{\zeta-\eta}\,d\zeta \\
\varphi_\infty(\eta) &= (1-\eta)^{-x}\,e^{\frac{\eta t}{1-\eta}} \\
f(\mu,z) &= \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1-\tau^k\mu}\,z^k
\end{aligned}$$

The kernel $J(\eta,\eta')$, which acts on a circle centered at 0 with radius less than τ, is analyzed by the steepest descent method.

Note: m now appears inside the kernel!

Main Result: Step I.C.

We set

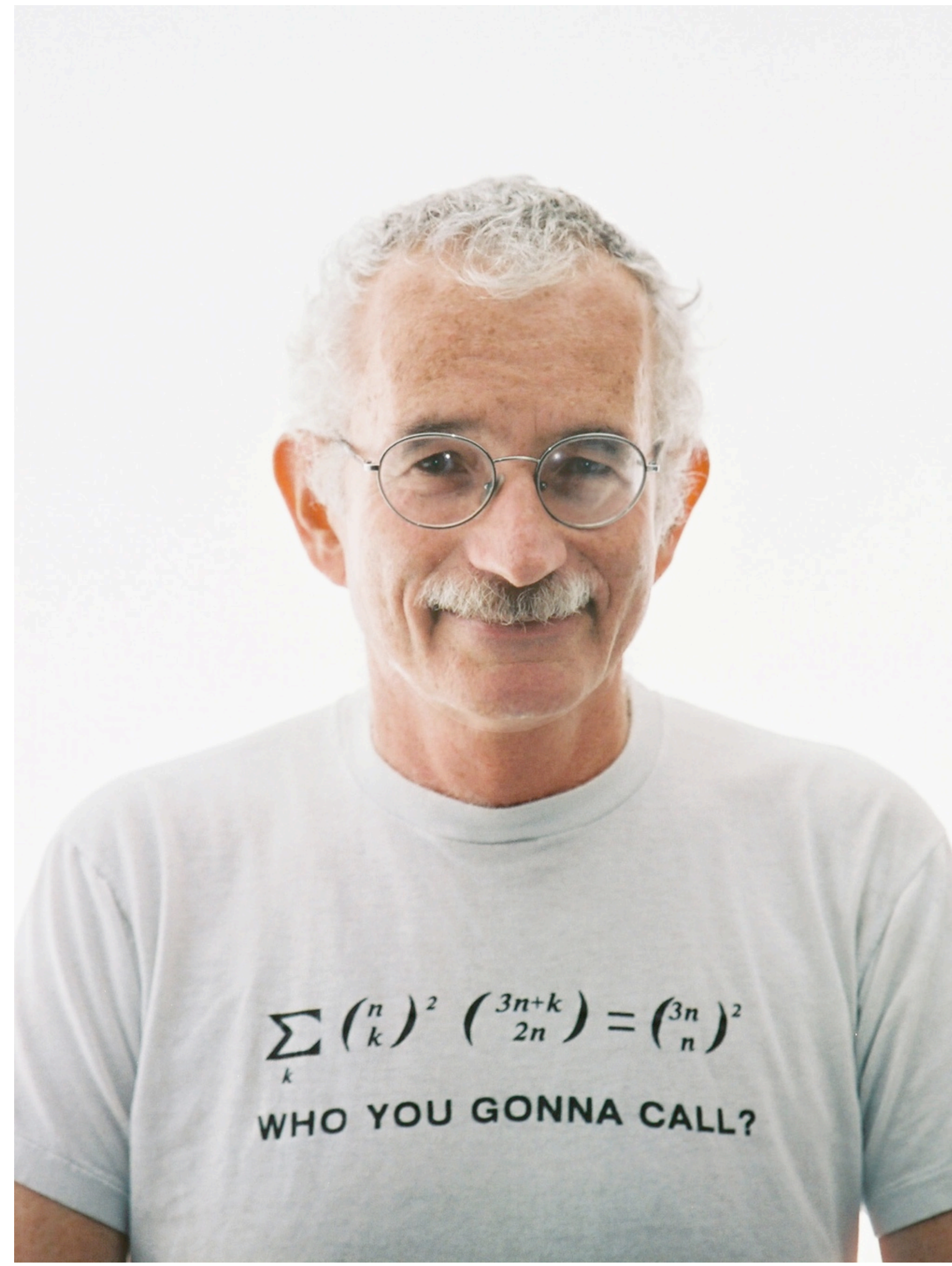
$$\sigma = \frac{m}{t}, c_1 = -1+2\sqrt{\sigma}, c_2 = \sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}, \gamma = q-p$$

Theorem (TW). When $0 \leq p < q$ we have

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_m(t/\gamma) - c_1 t}{c_2 t^{1/3}} \leq s\right) = F_2(s)$$

Theorem also has a current fluctuation formulation

Thanks to A. Schilling & D. Zeilberger for advice with the combinatorial identities



Thank you
for your attention