185B Homework 5

Question 1 Show that

$$\sum_{n=0}^{N} \left(\begin{array}{c} N \\ n \end{array} \right) \frac{n!}{N^n} \overset{N\to\infty}{\sim} \sqrt{\frac{\pi N}{2}}\,.$$

Question 2 What is the radius of convergence of the series

$$f(z) = 1 + z + z^2 + z^3 + \ldots ?$$

Obtain a power series for $f(z)$ valid about $z = -1/2$ and compute its radius of convergence. Draw a picture explaining what has happened.

Question 3 *Euler Transformation:* What is the radius of convergence of the power series

$$\log(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 \ldots ?$$

Consider a new variable

$$w = \frac{z}{1+z}\,.$$

Compute the power series of the logarithm in powers of w at the point corresponding to $z = 0$. In the w-complex plane what is the radius of convergence? For which values of z does the new series give an expression for the logarithm?

Question 4 Find all poles of the Gamma function $\Gamma(z)$ and compute their residues.

Question 5 *Natural boundary:* Consider the power series

$$f(z) = z + z^2 + z^4 + z^8 + z^{16} + \ldots\,.$$

Compute its radius of convergence. Try to decide whether it is possible to continue the function beyond its radius of convergence[1].

[1]Hint: Show $f(z^2) = f(z) - z$. Now show that if $f'(1)$ does not exist, then neither does $f'(-1)$. Repeat the same argument for $f(z^4)$, *etc*....