185B Homework 8

Question 1 Use the chain rule and the Cauchy Riemann equations to show that $\varphi(w)$ harmonic implies $\varphi(f(z))$ is also harmonic for an analytic map $z \mapsto w = f(z)$. Now apply the same method to Helmholtz's equation $(\partial_u^2 + \partial_v^2)\varphi = \Lambda\varphi$ to show that it becomes $(\partial_x^2 + \partial_y^2)\varphi = \Lambda|dw/dz|^2\varphi$.

Question 2 Invert the mapping $z \mapsto w = \frac{1}{i}\frac{z-i}{z+i}$. Show that it maps the unit disc to the upper half plane.

Question 3 Perform the Poisson integral for the following boundary conditions

1. $\phi(e^{i\theta}) = 1$,
2. $\phi(e^{i\theta}) = \cos\theta$.

Check that your result in the interior of the disc is real and harmonic. Now introduce a unit vector $\vec{e}(\theta) = \hat{x}\cos\theta + \hat{y}\sin\theta$ and radial vector $\vec{r} = x\hat{x} + y\hat{y}$ with $r \equiv |\vec{r}|$. Show that the Poisson integral can be rewritten

$$\varphi(\vec{r}) = \frac{1}{2\pi}(1 - r^2)\int_0^{2\pi} \frac{d\theta\phi(\vec{e}(\theta))}{|\vec{r} - \vec{e}(\theta)|^2}\,.$$

Now generalize the Poisson integral to arbitrary dimensions.

Question 4 Let m be a positive integer. Verify

$$(1+h)^{\frac{1}{m}} = \sum_{n=0}^{\infty} \frac{\frac{1}{m}(\frac{1}{m}-1)\cdots(\frac{1}{m}-n+1)}{n!}\,h^n\,, \quad |h| < 1.$$

Question 5 Show that the following transformations preserve angles in $\mathbb{R}^n$:-

1. $\vec{x} \mapsto \vec{x}' = \vec{x} + \vec{a}$
2. $\vec{x} \mapsto \vec{x}' = \lambda\vec{x}$
3. $\vec{x} \mapsto \vec{x}' = O\vec{x}$ where $O^T O = I$
4. $\vec{x} \mapsto \vec{x}' = \frac{\vec{x} + |\vec{x}|^2\vec{b}}{1 + 2\vec{x}\cdot\vec{b} + |\vec{b}|^2|\vec{x}|^2}$.

The last map is called a special conformal transformation. It is best studied by relating it to the composition (inversion)$\circ$(translation)$\circ$(inversion) where inversions map $\vec{x} \mapsto \frac{\vec{x}}{|\vec{x}|^2}$. For each of the above transformations write down linear differential operators K such that $f(\vec{x}') - f(\vec{x}) = \epsilon K f(\vec{x}) + \mathcal{O}(\epsilon^2)$ where the $\epsilon = 0$ amounts to making no transformation at all (for example $\lambda = e^\epsilon$). Study commutators of these operators (*i.e.* $[K_1, K_2] \equiv K_1 K_2 - K_2 K_1$).

Question 6 Verify the group property for Möbius mappings. Show that Möbius mappings send generalized circles to generalized circles.