Q1 40 Last Initial C

Q2 30 Section 1

Q3 30 FULL Name Captain Conundrum

Σ 100

MIDTERM EXAMINATION

21A §A01-05, 1:10-2:00 pm

Wednesday January 30, 2008

Declaration of honesty: I, the undersigned, do hereby swear to uphold the highest standards of academic honesty, including, but not limited to, submitting work that is original, my own and unaided by notes, books, calculators, mobile phones, bread machines, artificial intelligence or any other electronic device.

Signature CC Date 1/8/30

Question 1 *(Two parts, total 40 points)*

(i) Write out in words what the following mathematical symbols mean and then give a precise definition:

$$\lim_{x \to -\infty} f(x) = L\,.$$

As x approaches minus infinity, $f(x)$ approaches L.

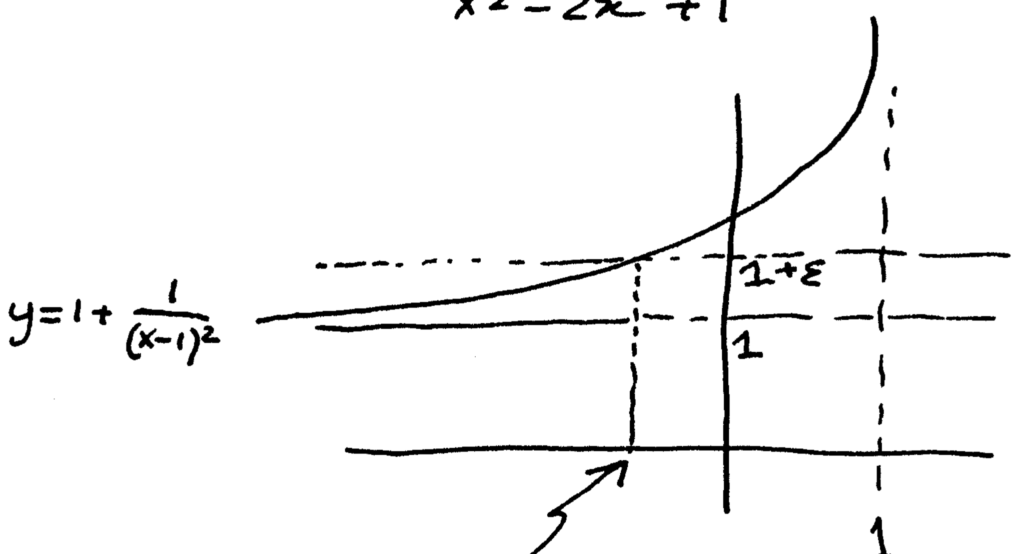
Def$^{\underline{n}}$ $\forall \varepsilon > 0 \quad \exists B < 0$ such that

$$x < B \Rightarrow |f(x) - L| < \varepsilon$$

(ii) Use your definition in (i) to *prove*

$$\lim_{x \to -\infty} \frac{x^2 - 2x + 2}{x^2 - 2x + 1} = 1\,.$$

Scratch: $\dfrac{x^2 - 2x + 2}{x^2 - 2x + 1} = 1 + \dfrac{1}{(x-1)^2}$



$1 - \dfrac{1}{\sqrt{\varepsilon}}$ This is our B

$\left(\begin{array}{l} 1 \text{ solved} \\ 1 + \frac{1}{(x-1)^2} = 1 + \varepsilon \end{array} \right)$

Working for question 1

Proof Let $\varepsilon > 0$

Suppose $x < 1 - \frac{1}{\sqrt{\varepsilon}}$

$\Rightarrow \quad x - 1 < -\frac{1}{\sqrt{\varepsilon}}$

$\Rightarrow \quad (x-1)^2 > \frac{1}{\varepsilon}$ $\quad$ (e.g. $-3 < -2$ $\Rightarrow$ $9 > 4$)

$\Rightarrow \quad \frac{1}{(x-1)^2} < \varepsilon$

$\Rightarrow \quad -\varepsilon < \frac{1}{(x-1)^2} < \varepsilon$

$\Rightarrow \quad 1 - \varepsilon < \frac{1}{(x-1)^2} + 1 < 1 + \varepsilon$

$\Rightarrow \quad \left| \frac{x^2 - 2x + 2}{x^2 - 2x + 1} - 1 \right| < \varepsilon$ $\quad$ QED

Question 2 *(Three parts, total 30 points)*

(i) Define what it means for $f(x)$ to be continuous at x_0.

x_0 is in the domain of $f(x)$

and

$$\lim_{x \to x_0} f(x) = f(x_0)$$

(Don't be picky about endpoints, "$\lim_{x \to x_0} f(x) = f(x_0)$" should get close to full credit)

(ii) Suppose x_0 is not in the domain of $f(x)$ but $\lim_{x \to x_0} f(x)$ exists. Explain how to define the continuous extension of $f(x)$ to x_0.

Define a new <u>continuous</u> function

$$g(x) = \begin{cases} \lim_{x \to x_0} f(x) & x = x_0 \\ f(x) & x \neq x_0 \end{cases}$$

This is the continuous extension of $f(x)$ to x_0

(iii) Find the continuous extension of

$$\frac{x^3 + 6x^2 + 11x + 6}{(x+1)(x+2)(x+3)}$$

to the points $x = -1$, $x = -2$ and $x = -3$.

Notice $\dfrac{x^3+6x^2+11x+6}{(x+1)(x+2)(x+3)}$

$$= \frac{(x+1)(x+2)(x+3)}{(x+1)(x+2)(x+3)}$$

$= 1$ whenever $x \neq -1, -2, -3$.

$\Rightarrow$ The continuous extension to $-1, -2, -3$

is $g(x) = 1$ ("canceling common factors")

Question 3 *(Three parts, total 30 points)* Find the following *three* limits

(i)

$$\lim_{\theta\to 0} \frac{\sin\sqrt{2}\,\theta}{\sqrt{2}\,\theta}$$

$$\lim_{\theta\to 0} \frac{\sin\sqrt{2}\theta}{\sqrt{2}\theta} = 1$$

(ii)

$$\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\,\theta}.$$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta} = \frac{1}{2}\lim_{\theta\to 0} \frac{2\theta\sin\theta}{\theta\sin 2\theta}$$

$$= \frac{1}{2}\lim_{\theta\to 0}\frac{2\theta}{\sin 2\theta}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$

$$= \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$$

(iii)

$$\lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to-\infty} \frac{\cancel{e^{-x}}(e^{2x}-1)}{\cancel{e^{-x}}(e^{2x}+1)}$$

$$= \frac{\lim_{x\to-\infty}(e^{2x}-1)}{\lim_{x\to-\infty}(e^{2x}+1)} = \frac{-1}{1} = -1$$