MAT 21A Review Session

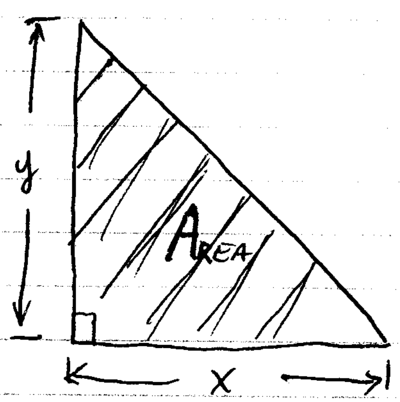
Final Exam: Monday, December 12, 1:30 p.m. SCI LEC 123

Extra Problems w/ Detailed Solutions: www.ecalculus.org

- Related Rates:

A particular right triangle has legs that are of different length. If the length of the short leg is increasing at the rate of 1 cm/sec and the length of the long leg is decreasing at the rate of 2 cm/sec, what is the rate of change of the area at the instant the short leg is 5 cm and the long leg is 12 cm?

Solution:



x = length of short leg

y = length of long leg

$\frac{dx}{dt} = 1$ cm/sec $\quad \frac{dy}{dt} = -2 \frac{cm}{sec}$

Want to find $\frac{dA}{dt}$ when $x = 5$ cm and $y = 12$ cm

Recall, $A = \frac{1}{2} x \cdot y$

So,

$$\frac{dA}{dt} = \frac{1}{2}\left[\frac{dx}{dt} y + x \frac{dy}{dt}\right]$$

$$= \frac{1}{2}\left[1 \cdot 12 + 5 \cdot -2\right] = \frac{1}{2}[2]$$

$$\boxed{\Rightarrow \frac{dA}{dt} = 1 \text{ cm}^2/\text{sec}}$$

MAT 21A REVIEW SESSION

- **LINEARIZATION and DIFFERENTIALS**

Use linear approximation to approximate $(122)^{2/3}$.
Hint: $(125)^{2/3} = [(125)^{1/3}]^2 = 5^2 = 25$

SOLUTION:
For a differentiable function f, at $x=a$, the approximating function
$L(x) = f(a) + f'(a)(x-a)$ is the linearization of f at a. The approximation
$$f(x) \approx L(x)$$
of f by L is THE standard linear approximation of f at a.
(See page. 241, Wehagi)

So, $f(x) = x^{2/3}$ and $f'(x) = \frac{2}{3} x^{-\frac{1}{3}}$.

Since we know $f(125) = 25$, let $a = 125$ and
$$f'(125) = \frac{2}{3}(125)^{-1/3} = \frac{2}{15}$$

Thus, $f(122) \approx L(125) = f(125) + f'(125)(122-125)$

$$\Rightarrow f(122) \approx 25 + \frac{2}{15}(-3) = 24\tfrac{3}{5} = 24.6$$

NOTE, ACTUAL VALUE OF $(122)^{2/3} \approx 24.5984$ so the linear approximation gives a relatively good result.

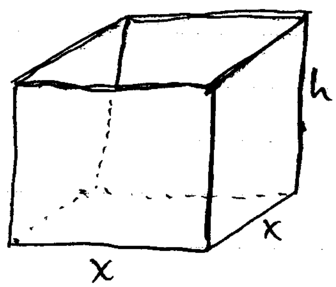
MAT 21A Review Session

- **Optimization:**

Of all the topless rectangular boxes with square bases that have a volume of $3\ ft^3$, what are the dimensions of the box which uses the least material?

Solution:

Make a sketch and label appropriately.



We want to minimize the surface area, S, where

$$S = x^2 + 4xh$$

and the constraint $x^2 h = 3\ ft^3$
$\Rightarrow h = 3/x^2$.

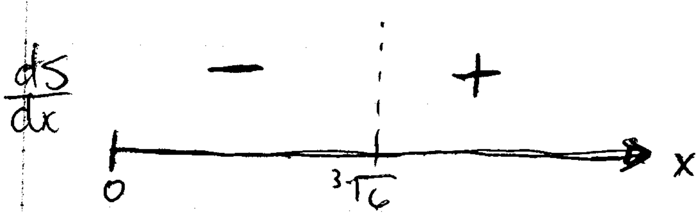
So, $S = x^2 + 4x\left(\frac{3}{x^2}\right) = x^2 + \frac{12}{x}$.

To find a minimum of S, take the derivative w.r.t. x

$\Rightarrow \frac{dS}{dx} = 2x - \frac{12}{x^2}$ and set equal to zero and solve for x.

$\Rightarrow 2x - \frac{12}{x^2} = 0 \Rightarrow x^3 = 6 \Rightarrow x = \sqrt[3]{6}\ ft.$

One should check that S has a minimum at $x = \sqrt[3]{6}$



Indeed, $x = \sqrt[3]{6}$ is a minimum. Thus, the dimensions of the box are: $\boxed{x = \sqrt[3]{6}\ ft,\ h = \frac{3}{6^{2/3}}\ ft}$

21A Practice Final Exam

By David Cherney

Name:
Student ID:
Section:
Sign here if you will not be cheating on this exam:_________________________

1a. Assume f(x) is a differentiable function over the domain(f). Define the derivative of f at x=a in terms of a limit.

1b. Assume g(x) is not differentiable anywhere on its domain. Evaluate

$$\lim_{b \to a} \frac{g(b) - g(a)}{b - a}$$

2. Differentiate following functions

$f\left(x\right) = 5xsin\left(3e^{6cos(4x)}\right)$

$g\left(x\right) = log_{(x^{-1})}\left(sinx\right)$

3. A speaker sends out a sound wave in the shape of a hemisphere. The speed of sound is 343m/s. At what rate is the surface area of the sound wave increasing when it is one meter from the speaker? Two meters? Two thousand meters? Two thousand and one meters?

4. Draw the following piece wise defined function and its derivative. List the x values at which the function is discontinuous or not differentiable. Do the same for the derivative function.

$f\left(x\right) = x^2$ for $-1 < x \leq 0$, $sin^2(x)$ for $0 < x \leq \pi$, $sin(x)$ for $\pi < x \leq 2\pi$

Solutions:

1a. Assume $f\left(x\right)$ is a differentiable function over $domain\left(f\right)$. Define the derivative of f at $x = a$ in terms of a limit.

Soln: Either of the following are fine:

$f'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$ or $f'(x) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$

1b. Assume g(x) is not differentiable anywhere on its domain. Evaluate

$\lim_{b \to a} \frac{g(b) - g(a)}{b - a}$

Soln: Well, this is the definition of the derivative of g at a according to the answer to 1a. But we are told g is not differentiable anywhere, so its certainly not differentiable at a. In other words the derivative does not exist. That means the limit does not exist. This was one of those questions to see if you understood what was going on in an earlier problem, cuz maybe you just had some symbols memorized. Professors love these.

2. Differentiate following functions

$f(x) = 5xsin\left(3e^{6cos(4x)}\right)$

Soln: This is quite a function, but you should be able to differentiate just about any function you can read with the tools you picked up in this class.
$f'(x) = 5sin\left(3e^{6cos(4x)}\right) + 5xcos\left(3e^{6cos(4x)}\right)3e^{6cos(4x)}(-6sin(4x))4$
If you can't just do it by sight, do some substitutions. For instance you could substitute
$g = 4x$, $h = 6cos(g)$, $I = 3e^h$
Then $f(x) = 5xsin(I)$ and just the product and chain rule are needed.
$f'(x) = 5sin(I) + 5xcos(I)I'$
$f'(x) = 5sin\left(3e^h\right) + 5xcos\left(3e^h\right)3e^h h'$
$f'(x) = 5sin\left(3e^{6cos(g)}\right) + 5xcos\left(3e^{6cos(g)}\right)3e^{6cos(g)}(-6sin(g))g'$
$f'(x) = 5sin\left(3e^{6cos(4x)}\right) + 5xcos\left(3e^{6cos(4x)}\right)3e^{6cos(4x)}(-6sin(4x))4$
Don't wast time simplifying; we know you can simplify; you would not have made it this far if you couldn't.
NEXT!

$g(x) = log_{(x^{-1})}(sinx)$

Soln: This isn't as complicated, it just... weird. The base of the log is a variable! Sometimes you have to manipulate a function a bit before you see how to go about differentiating it. The simplest thing to do here, I think, is to use the change of base formula for logs:
$log_a b = \frac{log_c b}{log_c a}$ where c is any number you want.
Natural logs are nice when differentiating, so lets choose $c = e$.

$g(x) = \frac{ln(sinx)}{ln(x^{-1})} = -\frac{ln(sinx)}{ln(x)}$
Now just use the quotient rule. Lets see.. low-d-high minus high-d-low over low low...
$g'(x) = -\left(\frac{ln(x)\frac{1}{sinx}cosx - ln(sinx)\frac{1}{x}}{(ln(x))^2}\right)$
If you forgot your log properties, time to make some flash cards. And while I'm on the topic here are some other things you might want to commit to memory: trig identities, derivatives of all 6 trig funtions and all 6 of their inverses, all the differentiation rules...

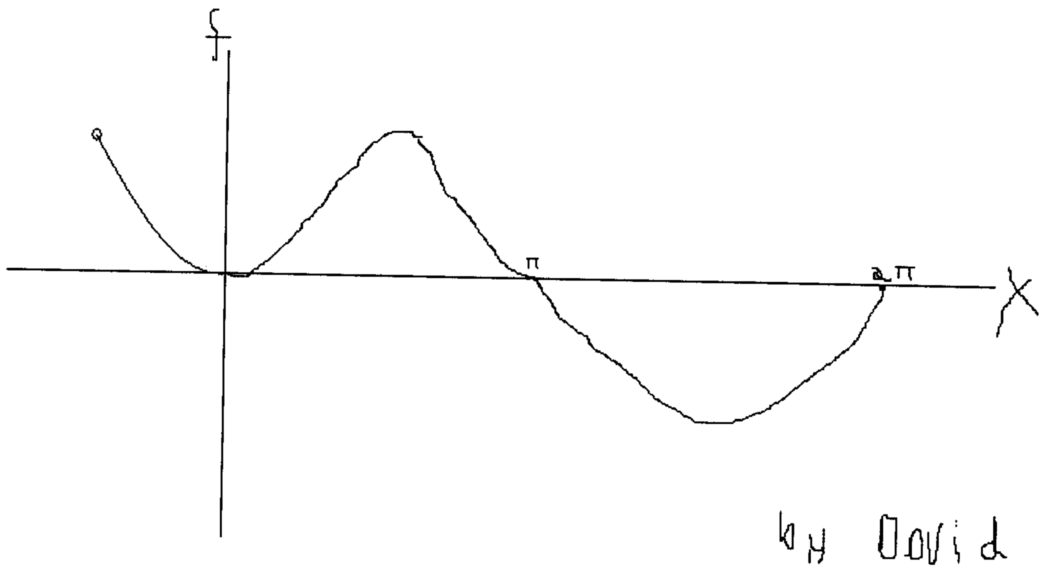
3. A speaker sends out a sound wave in the shape of a hemisphere. The speed of sound is 343m/s. At what rate is the surface area of the sound wave increasing when it is one meter from the speaker? Two meters? Two thousand meters? Two thousand and one meters?

I love question that require you to picture invisable things! OK, a hemisphere is half of a sphere so its surface area is half that of a sphere. (hemispheres have a circular face, but the sound wave is not occupying that area, so ignore it.)
$A = \frac{1}{2}4\pi r^2$
How do we get an equation involving the time derivative of the surface area?
$\frac{dA}{dt} = \frac{1}{2}4 \cdot 2\pi r\frac{dr}{dt}$
So to find the time derivative of the surface area we need to know the rate of change of the radius of the hemisphere $\frac{dr}{dt}$ and the radius r. We have both: the speed of sound and the given radii respectively.
$\frac{dA}{dt} = 4\pi \cdot 1m \cdot 343\frac{m}{s} = 4\pi \cdot 1 \cdot 343\frac{m^2}{s}$ I love it when I get the right units.
$\frac{dA}{dt} = 4\pi \cdot 2 \cdot 343\frac{m^2}{s}$
$\frac{dA}{dt} = 4\pi \cdot 2000 \cdot 343\frac{m^2}{s}$
$\frac{dA}{dt} = 4\pi \cdot 2001 \cdot 343\frac{m^2}{s}$
See, who needs a calculator?

4. Draw the following piece wise defined function and its derivative. List the x values at which the function is discontinuous or not differentiable. Do the same for the derivative function.

$f(x) = x^2$ for $-1 < x \leq 0$, $sin^2(x)$ for $0 < x \leq \pi$, $sin(x)$ for $\pi < x \leq 2\pi$

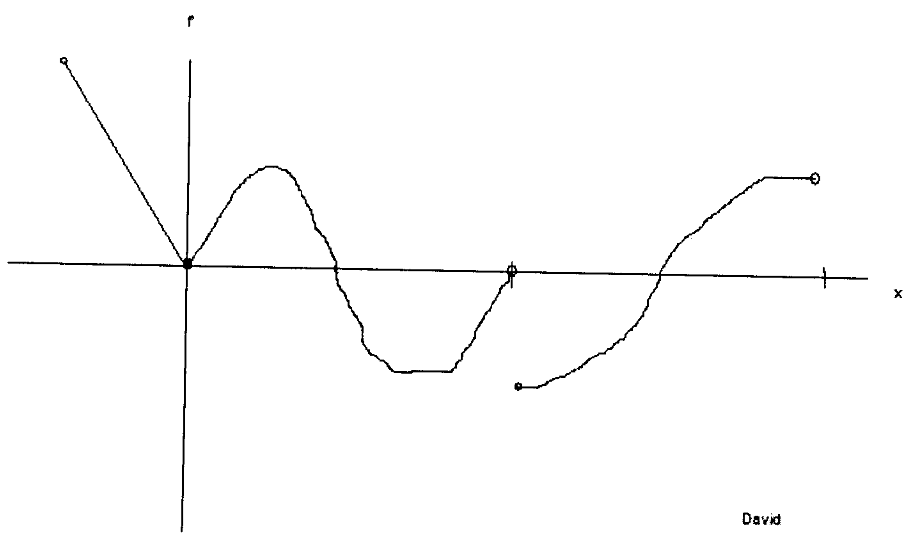
Alright, bear with me, I'm using msPaint... on a touch pad... on my lap... at midnight... on an empty stomach and... There is no excuse actually, just know you are laughing WITH me. Here goes:



The only place on the $domain\,(f)$ where f is not continuous is at $x = 2\pi$.
Why? Remember that a function is continous at a if
$\lim_{x \to a} f(x) = f(a)$
and that a limit exists if and only if the right and left limit exist. The function is not defined for $x > 2\pi$, so the limit can not be evaluated from the right.
As for differentiability, we know
x^2 is differentiable for $-1 < x < 0$,
$sin^2(x)$ is differentiable for $0 < x < \pi$, and
$sin(x)$is differentiable for $\pi < x < 2\pi$.
So we only need to wory about $x = 0, \pi$ and 2π.
The definition of the derivative of f at x is
$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
At $x = 2\pi$ we can't evaluate this limit from the right because $f(2\pi + h)$ has no meaning for $h > 0$, so the derivative there does not exist.
At $x = \pi$ the limit from the right is -1 but he limit from the left is 0 so the derivative does not exist there.
At $x = 0$ the limit from the the left and right are 0, so the derivative does exist there. Wow, a piece-wise defined function can be continuous where the

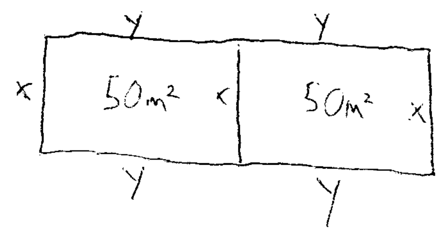
glue sits!

Now the derivative function. There are a few things you could do to sketch the derivative curve: 1)qualitatively sketch it by looking at the slope of the function at each location or 2) Actually calculate then plot the derivative function in each reagon. You did 1 on the midterm, since we are given more information here, namely we have an explicit function, lets do 1. You will need one of those trig identities which belong on yer flashcards.
$2x$ for $-1 < x < 0$,
$2sin(x)cos(x) = sin(2x)$ for $0 < x < \pi$, and
$cos(x)$for $\pi < x < 2\pi$.



This curve is continuous on $(-1, \pi) \bigcup (\pi, 2\pi)$ and discontinuous at π.
It is differentiable on $(-1, 0) \bigcup (0, \pi) \bigcup (\pi, 2\pi)$ and not differentiable at 0, π or 2π.
Hope this helps, good luck on the real deal!
-DMC

Problem: Find the least amount of fencing required to enclose two equal sized rectangular areas sharing a common side, as shown in the picture, if the total area is to be 100 m^2.



Solution:

The total amount of fencing needed is $3x+4y$ meters. Since the area is given by $A=50=xy$, we see that $\frac{50}{x}=y$. Substituting in to the fencing equation gives $F=3x+4y=3x+4\left(\frac{50}{x}\right)=3x+\frac{200}{x}$. We now have the required amout of fencing in an equation in a single varidble, so we can differentiate and set it equal to zero to find the maximum:

$$F'(x)=3+(-1)\frac{200}{x^2}=0$$
$$3=\frac{200}{x^2}$$
$$x^2=\frac{200}{3}$$
$$x=\pm\sqrt{\frac{200}{3}}.$$

The negative root is absurd, so we disregard it. We can then solve for y as well, and $y=\frac{50}{x}=\frac{50}{\sqrt{\frac{200}{3}}}=\frac{5\sqrt{3}}{\sqrt{2}}$. We then have that

$$F\left(\sqrt{\tfrac{200}{3}}\right)=3\left(\frac{\sqrt{200}}{\sqrt{3}}\right)+4\left(\frac{5\sqrt{3}}{\sqrt{2}}\right)=10\sqrt{6}+10\sqrt{6}=20\sqrt{6}.$$

Problem: Prove that $\lim_{x\to\infty} \frac{1}{x^2} = 0$.

Solution:

We want to show that $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ such that $x > N$ implies $\left|\frac{1}{x^2} - 0\right| < \varepsilon$. First we must find an N given some ε. We want $\left|\frac{1}{x^2} - 0\right| < \varepsilon$, and $\left|\frac{1}{x^2} - 0\right| = \left|\frac{1}{x^2}\right| = \frac{1}{x^2}$, so we want $\frac{1}{x^2} < \varepsilon$, or equivalently $\frac{1}{\varepsilon} < x^2$, or $\sqrt{\frac{1}{\varepsilon}} < x$. Thus let $N = \sqrt{1/\varepsilon}$.

Now if $x > N$, $x > \sqrt{1/\varepsilon}$, so $x^2 > \frac{1}{\varepsilon}$ and $\varepsilon > \frac{1}{x^2}$. Then $\left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2} < \varepsilon$, which was exactly what we wanted to show. Q.E.D.