HW 9

Q1 Let $S_N = \sum_{n=1}^{N} \frac{(-)^{n+1}}{n^2}$ alternating

$$S_5 = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} = \frac{3019}{3600}$$

$$S_6 = S_5 - \frac{1}{36} = \frac{973}{1200}$$

$$\frac{S_5 + S_6}{2} = \frac{2969}{3600}$$

$$\Rightarrow S_\infty = \frac{2969}{3600} \pm \frac{1}{2}\left(\frac{3019}{3600} - \frac{973}{1200}\right)$$

$$\simeq .82 \pm .01 \ \text{(th.)}$$

Positive terms

Let $S_N = \sum_{n=1}^{N} \frac{1}{n^2}$

$$S_6 = \frac{5369}{3600} = 1.4913888\ldots \quad \text{while } \frac{\pi^2}{6} = 1.6449\ldots$$

crumby

But we estimate error

$$\int_7^\infty \frac{dx}{x^2} < S_\infty - S_6 < \int_6^\infty \frac{dx}{x^2} \Rightarrow S_\infty = 1.64 \pm .01 \ \text{(th.)}$$

$$\int_7^\infty \frac{dx}{x^2} = .14\ldots \qquad \int_6^\infty \frac{dx}{x^2} = .16$$

integral test helps a lot!

Q2 §10.6

q2 $a_n = \frac{(n+1)^2}{n \cdot 2^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+2)^2}{(n+1)2^{n+1}}}{\frac{(n+1)^2}{n 2^n}} = \lim_{n\to\infty} \frac{n(n+2)^2}{2(n+1)^3} = \frac{1}{2} < 1$$

$\Rightarrow \sum a_n$ converges

Q7 $a_n = \frac{n!}{3^n}$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!/3^{n+1}}{n!/3^n}$

$$= \lim_{n\to\infty} \frac{n+1}{3} = \infty$$

$\Rightarrow \sum a_n$ diverges

Q8 a) $a_n = \frac{2^n}{n^5}$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)^5}{2^n/n^5}$

$$= \lim_{n\to\infty} \frac{2(n+1)^5}{n^5} = 2 > 1$$

$\Rightarrow \sum a_n$ diverges

b) $\lim_{n\to\infty} \frac{2^n}{n^5} = \infty$ also diverges by nth term test

2.12 $\quad a_n = \dfrac{2^n x^n}{n!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1} x^{n+1}/(n+1)!}{2^n x^n / n!}$$

$$= \lim_{n\to\infty} \frac{2x}{n+1} = 0 \quad \forall x$$

series converges $\forall x$ (for $-$ve x just use for absolute convergence)

__Indeed__ $\displaystyle\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = e^{2x}$

2.13 $\quad a_n = \dfrac{n^n}{3^{n^2}}$

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(\frac{n}{3^n}\right)^n\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(\frac{n}{3^n}\right) = 0 < 1$$

$\Rightarrow \sum_n a_n$ __converges__

2.14 $\quad a_n = \dfrac{(1+\frac{1}{n})^n (2n+1)^n}{(3n+1)^n}$

$$\Rightarrow \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})(2n+1)}{3n+1} = \frac{2}{3} < 1$$

$\Rightarrow \sum_n a_n$ __converges__

$2.14 \quad a_n = \frac{\log n}{n^2}$

use $\int \frac{dx \log x}{x^2} = -\frac{\log x + 1}{x}$

2.20 . could use integral test, quicker:

$$\frac{n}{n^3+1} = \frac{1}{\frac{1}{n}(n^3+1)} = \frac{1}{n^2+\frac{1}{n}} < \frac{1}{n^2}$$

now use integral test here

2.26 a) $p_n = \frac{1}{n} \quad \lim_{n\to\infty} \frac{p_{n+1}}{p_n} = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = 1$

but $\sum \frac{1}{n}$ diverges

b) $p_n = \frac{1}{n^2} \quad \lim_{n\to\infty} \frac{p_{n+1}}{p_n} = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = 1$

but $\sum \frac{1}{n^2}$ converges

oops ratio test!

nth root test:

a) $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = ?$. Easier $\lim_{n\to\infty} \log \left(\frac{1}{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{n} = 0$

$= \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = \log 0 = 1$ but $\sum \frac{1}{n}$ diverges

b) similar to a)

11.9 #15. ?

q 2 $-\frac{1}{1+\frac{1}{2}}+\frac{1}{1+\frac{1}{4}}-\frac{1}{1+\frac{1}{8}}\cdots = \sum_{n=1}^{\infty}(-)^n a_n$

$a_n = \frac{1}{1+\frac{1}{2^n}}$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{1+\frac{1}{2^n}} = 1 \neq 0$

diverges by nth term test

q 6 $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \sum_{n=1}^{\infty}(-)^{n+1}a_n$

$a_n = \frac{1}{n^2}$ $\lim_{n\to\infty}\frac{1}{n^2}=0$

converges by alternating series test.

q 10 $\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\cdot 2^{(-n)} = \log 3/2$

q 12 $\sum_{n=1}^{\infty}(-)^n \log\frac{1}{n}$. diverges because

$\lim_{n\to\infty}\log\frac{1}{n} = \log 0 = 1 \neq 0$

— nth term test.

q 20 $\sum_{n=1}^{\infty}\frac{\sin n}{n^{1.01}}$ converges absolutely

comparison with $p=1.01$ series.

26 $\displaystyle\sum_{n=1}^{\infty} \frac{(-\pi)^{2n+1}}{(2n+1)!}$ converges absolutely

(by ratio test)

32 a) $\frac{P}{Q} \xrightarrow[n\to\infty]{} 0$ ~~when~~ $\deg(P) < \deg(Q)$

Let $\zeta(p)$

$p>1$ by integral test

$$\int_1^{\infty} \frac{dx}{x^p} < \zeta(p) < 1 + \int_1^{\infty} \frac{dx}{x^p}$$

$$\frac{1}{p-1} \qquad\qquad \frac{p}{p-1}$$

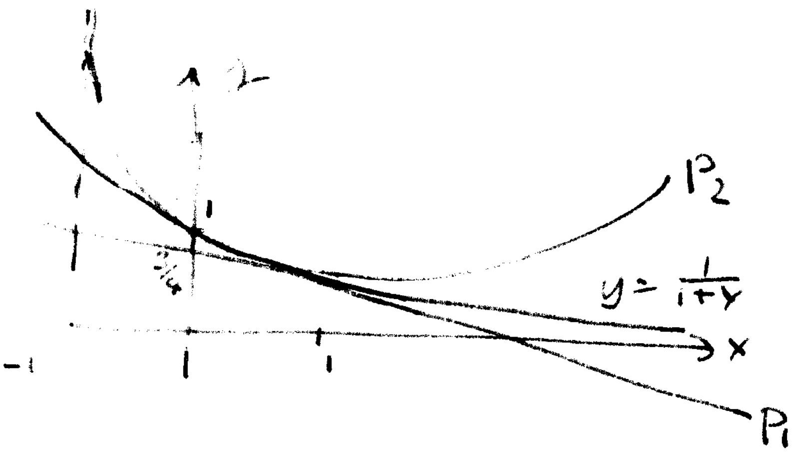
$$\Rightarrow \quad 1 < (p-1)\zeta(p) < p \quad \xRightarrow{p\to 1} \quad (p-1)\zeta(p) \longrightarrow 1$$

Q. 2

$$\frac{1}{1+x} = \underbrace{\overbrace{\frac{1}{2} + \left.\frac{-1}{(1+x)^2}\right|_{x=1} (x-1)}^{P_1} + \frac{1}{2!}\left.\frac{2}{(1+x)^3}\right|_{x=1} (x-1)^2}_{P_2} + \cdots$$

$$P_1 = \frac{1}{2} - \frac{1}{4}(x-1) = 3/4 - \frac{1}{4}x$$

$$P_2 = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 = \frac{7}{8} - \frac{1}{2}x + \frac{1}{8}x^2$$



<u>Asymptotic</u>

at $x = 0$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

#2. $P_4(x) = (1+x)^4$ Taylor polynomial
<u>is</u> the polynomial!!

#3. <u>No</u> compare derivatives

#4. recall

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + x^n$$

<u>Binomial theorem</u>.