Q1 45

Q2 20

Q3 35

Σ 100

Last Initial C

Section A05

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FULL Name Captain Conundrum

MIDTERM EXAMINATION II

21D §A01-04, 9:00-9:50 am

Wednesday May 11, 2009

Declaration of honesty: I, the undersigned, do hereby swear to uphold the very highest standards of academic honesty, including, but not limited to, submitting work that is original, my own and unaided by notes, peeking at the person next to me whose answer is probably wrong anyway, books, calculators, mobile phones, immobile phones, twitters, blackberries, blueberries, boysenberries, raspberries, artificial intelligence or any other electronic device.

Signature CC **Date** 5/11/9

Question 1 *45 points* **[This is a multipart question – read carefully!]**
(i) Write an explicit formula for the vector field $\vec{F}(x, y)$ in the (x, y)-plane whose magnitude at the point P equals its distance from the origin O, whose direction is always perpendicular to the vector $\overrightarrow{OP}$ and whose cross product with the unit vector **k** along the z-axis points away from the origin. (ii) Sketch the vector field $\vec{F}$ in the (x, y)-plane. (iii) Determine whether $\vec{F}$ is conservative. (iv) Calculate the divergence of $\vec{F}$. (v) What is the flux of the field $\vec{F}$ around *any* closed path in the (x, y)-plane? Explain! [*If and only if* you cannot do part (i) *pretend* $\vec{F} = xy\mathbf{i} - \frac{1}{2}y^2\mathbf{j}$ to get partial credit for (ii-v)]

(i) Let $P = (x, y)$ then $|\overrightarrow{OP}| = \sqrt{x^2+y^2}$

$\overrightarrow{OP} = xi + yj$ so $yi - xj \perp \overrightarrow{OP}$

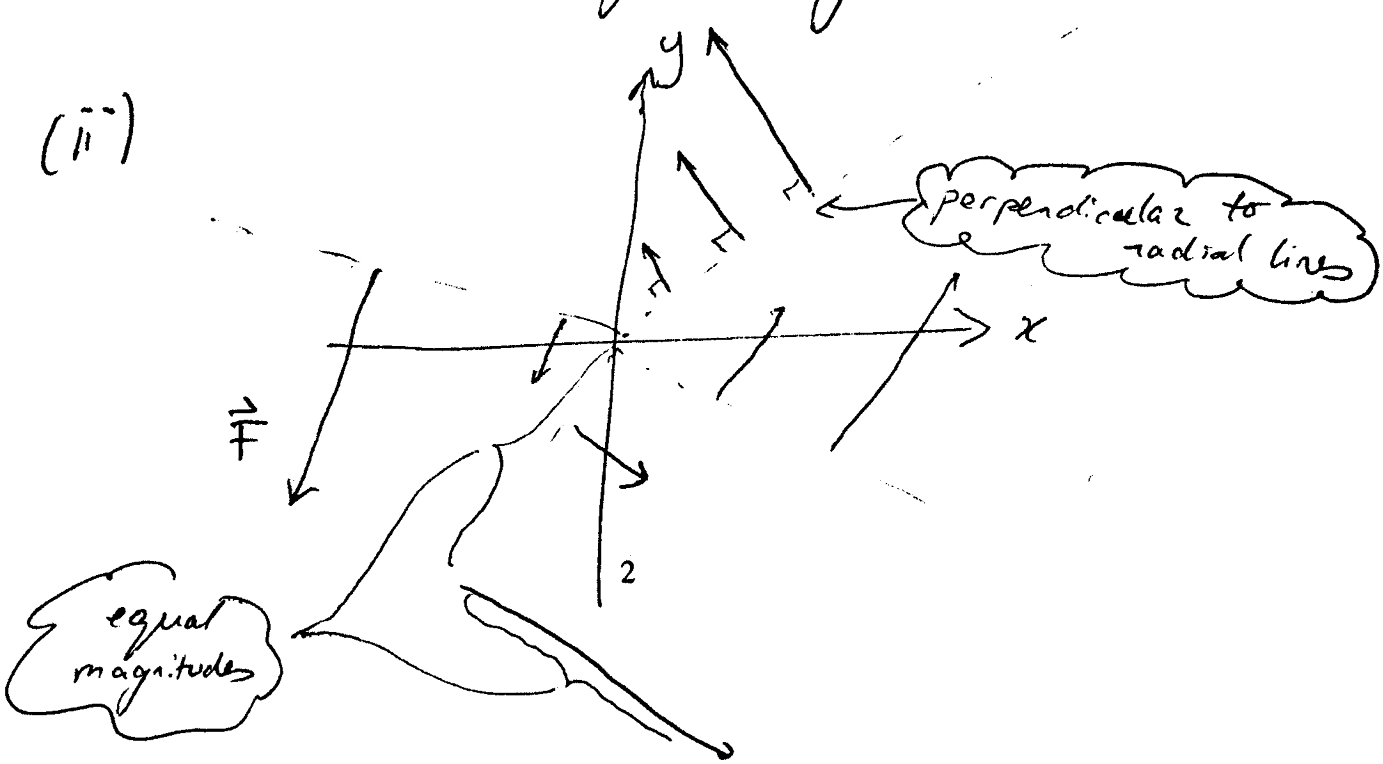
but $(yi - xj) \times k = -jy - ix = -\overrightarrow{OP}$

points out, but at least $|yi - xj| = \sqrt{x^2+y^2}$

Hence

$$\vec{F} = -yi + xj$$

(ii)



Working for question 1

(iii) $$\vec{F} = -y\,i + x\,j \overset{?}{\neq} \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

But $$\frac{\partial}{\partial y}(-y) = -1 \neq \frac{\partial}{\partial x}(x) = +1$$

$\vec{F}$ <u>NOT</u> conservative

(iv) $$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$$

(v) Use Green's Theorem

$$\text{FLUX} = \int_{C=\partial R} \vec{F}\cdot\vec{n}\,ds = \int_R dA \operatorname{div}\vec{F}$$

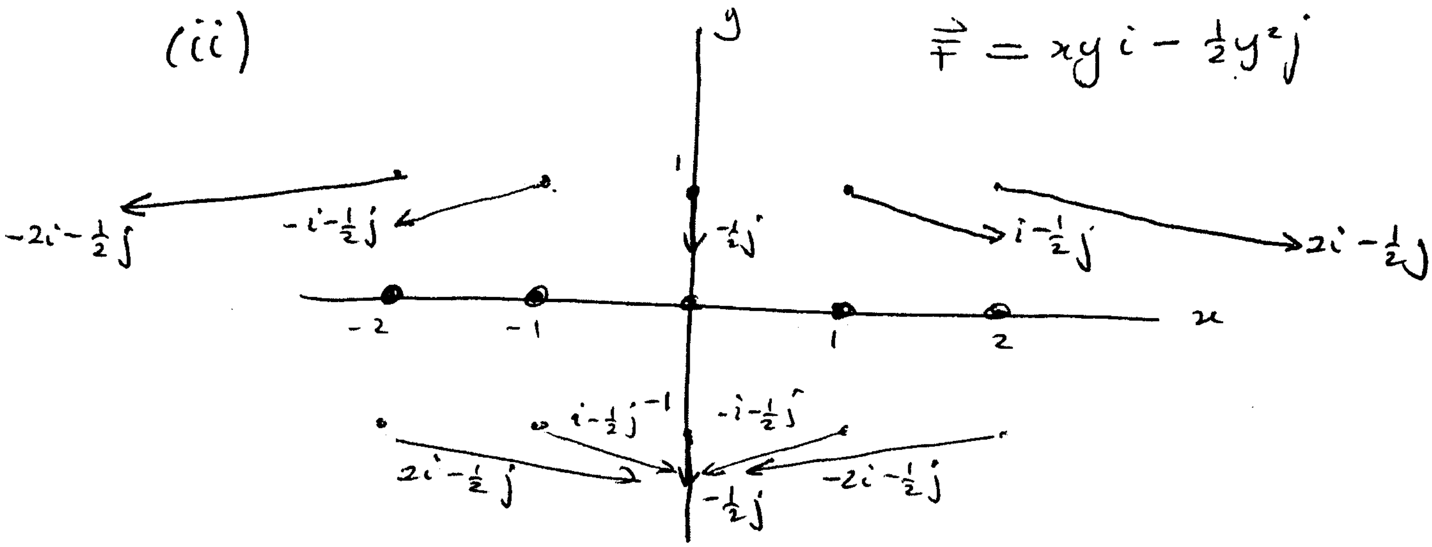
$$= \int_R dA \, . \, 0 = 0$$

vanishes because the <u>flux density</u> $\operatorname{div}\vec{F}$ vanishes everywhere.

Working for question 1 Less than $\frac{1}{2}$ total credit for the following:

Alternate answers

(ii) $\vec{F} = xy\,i - \frac{1}{2}y^2 j$



(iii) $\frac{\partial}{\partial y}(xy) = x \neq \frac{\partial}{\partial x}\left(-\frac{1}{2}y^2\right) = 0$

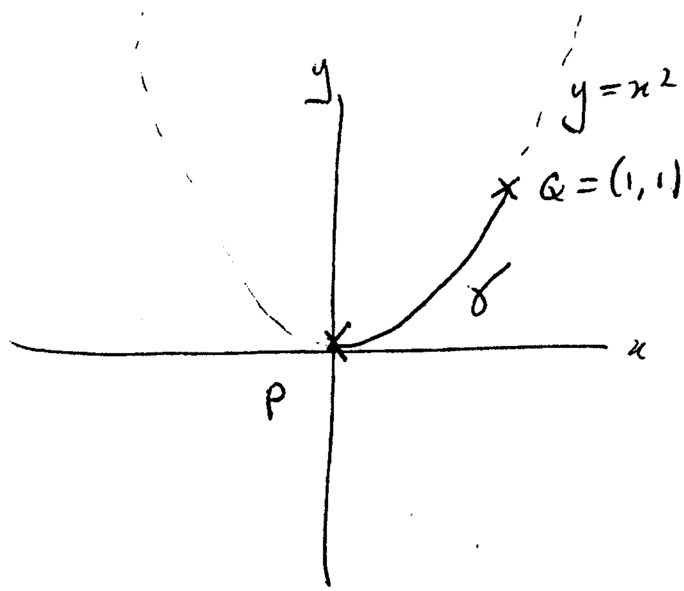
NOT CONSERVATIVE

(iv) $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(xy) - \frac{1}{2}\frac{\partial}{\partial y}y^2 = y - y = 0$

(v) AS FOR (v) ABOVE!

Question 2 *20 points*
Write down two different parameterizations for the path along the parabola $y = x^2$ in the (x, y)-plane beginning at $P = (0, 0)$ and ending at $Q = (1, 1)$. Use one of your parameterizations to compute the work done moving a particle along this path through the force field

$$\vec{F} = (x - y)\mathbf{i} + y\mathbf{j}\,.$$



First parameterization

$$\vec{r} = t i + t^2 j \qquad 0 \le t \le 1$$

Second parameterization

$$\vec{r} = t^2 i + t^4 j \qquad 0 \le t \le 1$$

something like

$$\vec{r} = (t-1) i + (t-1)^2 j \qquad 1 \le t \le 2$$

is also fine. There are <u>infinitely</u> <u>many</u> correct answers!

Working for question 2

$$\text{Work} = \int_{\gamma} \vec{F} \cdot d\vec{r}$$

$$= \int_{\gamma} \left[(x-y)\,dx + y\,dy \right]$$

$$= \int_0^1 \left[(t-t^2)\,d(t) + t^2\,d(t^2) \right]$$

$$= \int_0^1 \left[t\,dt - t^2\,dt + 2t^3\,dt \right]$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{2}{4} = \frac{2}{3}$$

Question 3 *35 points*

Gravitational field

(i) Find a potential function for the gravitational field

$$\vec{F} = -GmM\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} \qquad (G,\ m, \text{ and } M \text{ are constants}).$$

(ii) Let P_1 and P_2 be points at distance s_1 and s_2 from the origin. Show that the work done by the gravitational field in part (i) in moving a particle from P_1 to P_2 is

$$GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right).$$

(i) We did this in class so if you know already that $\vec{F} = \nabla U$ with

$$U = \frac{GmM}{\sqrt{x^2+y^2+z^2}}$$

this is OK so long as you check that its correct by direct computation

$$\nabla U = i\frac{\partial}{\partial x}\frac{GmM}{\sqrt{x^2+y^2+z^2}} + j\frac{\partial}{\partial y}\frac{GmM}{\sqrt{x^2+y^2+z^2}} + k\frac{\partial}{\partial z}\frac{GmM}{\sqrt{x^2+y^2+z^2}}$$

$$= GmM\left\{\frac{\cancel{2}ix\cdot\left(-\cancel{\tfrac{1}{2}}\right)}{(x^2+y^2+z^2)^{3/2}} + j(x\leftrightarrow y) + k(x\mapsto z)\right\}$$

$$= -\frac{GmM\,(xi+yj+zk)}{(x^2+y^2+z^2)^{3/2}} = \vec{F}$$

Working for question 3

Alternatively we need to solve

$$\frac{\partial U}{\partial x} = \frac{GmMx}{(x^2+y^2+z^2)^{3/2}} \Rightarrow U = \frac{-GmM}{(x^2+y^2+z^2)^{\frac{1}{2}}} + g(y,z)$$

use $\int \frac{x\,dx}{(x^2+y^2+z^2)^{3/2}} = \frac{1}{2}\int \frac{d x^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} = -2\cdot\frac{1}{2}\frac{1}{(x^2+y^2+z^2)^{1/2}} + C$

$$\Rightarrow \frac{\partial U}{\partial y} = \frac{GmMy}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial y} = \frac{GmMy}{(x^2+y^2+z^2} \Rightarrow g \text{ is } y \text{ independent}$$

similarly $\frac{\partial U}{\partial z}$ says g is x independent so g is a constant.

NB Assign a small (≤ 5) number of points to show $\vec{F}$ is actually conservative (this was not asked for).

(ii) Since $\vec{F}$ has a potential we can evaluate the work $= \int_{P_1}^{P_2} \vec{F}\cdot d\vec{r}$ by evaluating the potential at the endpoints

$$\text{work} = \frac{GmM}{S_2} - \frac{GmM}{S_1}$$

using that $\sqrt{x^2+y^2+z^2}$ is the distance from the origin