Q1 30 **Last Initial** C

Q2 40 **Section** A03

(A01 Port, A02 Porter, A03 Clement, A04 Barrette)

Q3 30

Σ 100 **FULL Name** CAPTAIN CONUNDRUM

MIDTERM EXAMINATION I

21D §A01-04, 9:00-9:50 am

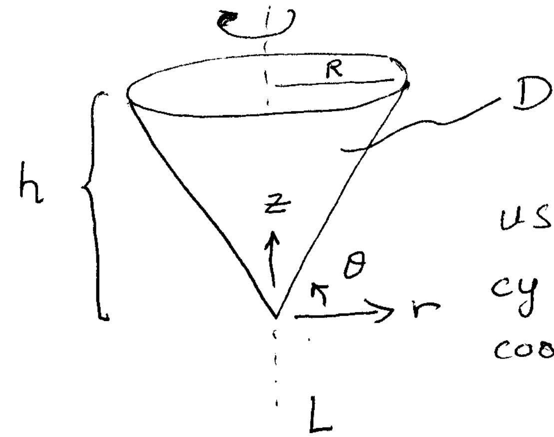
Friday April 17, 2009

Declaration of honesty: I, the undersigned, do hereby swear to uphold the very highest standards of academic honesty, including, but not limited to, submitting work that is original, my own and unaided by notes, peeking at the person next to me whose answer is probably wrong anyway, books, calculators, mobile phones, blackberries, blueberries, boysenberries, raspberries, artificial intelligence or any other electronic device.

Signature __________ **Date** 4/17/9

Question 1 *30 points*

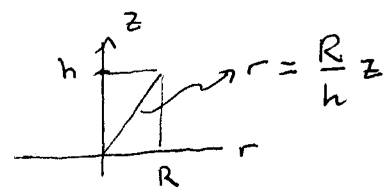
Find the moment of inertial of a uniform density cone of height h, mass M and radius R about its axis of symmetry.



use cylindrical coordinates (r, θ, z)

$$\text{Volume of cone} = \frac{1}{3}(\text{base area})(\text{height})$$

$$= \frac{1}{3}\pi R^2 h$$



$$\text{Density } \rho = \frac{M}{\frac{1}{3}\pi R^2 h}$$

Moment of inertia

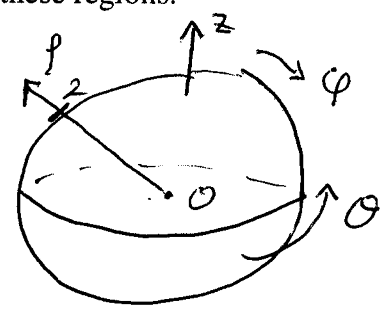
$$I_L = \int_D r^2 \rho \, dV = \int_0^h dz \int_0^{\frac{R}{h}z} r\,dr \int_0^{2\pi} d\theta \, \frac{Mr^2}{\frac{1}{3}\pi R^2 h}$$

$$= \int_0^h dz \left[\frac{1}{4} r^4\right]_0^{\frac{R}{h}z} 2\pi \frac{M}{\frac{1}{3}\pi R^2 h}$$

$$= \frac{3M}{2R^2 h}\left[\frac{1}{5} z^5\right]_0^h \left(\frac{R}{h}\right)^4 = \frac{3}{10} M R^2$$

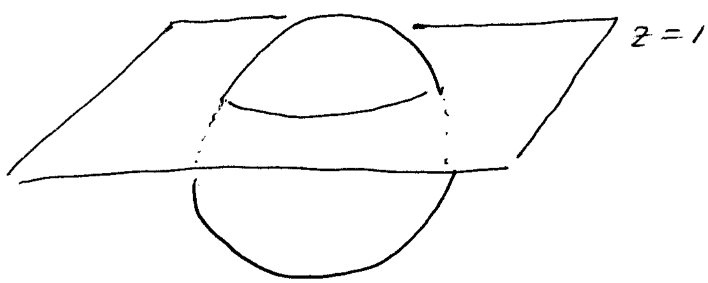
Question 2 *40 points*
Let (ρ, ϕ, θ) be spherical coordinates described in the standard way with respect to the z-axis. (i) Describe the domain bounded by $\rho \leq 2$. (ii) Sketch the regions obtained when this domain is cut by the plane $z = 1$. (iii) Find the volume of the smaller of these regions.

(i)



$\rho \leq 2$ is a <u>sphere</u>

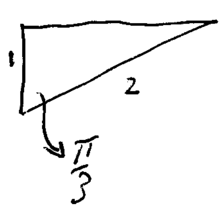
(ii)

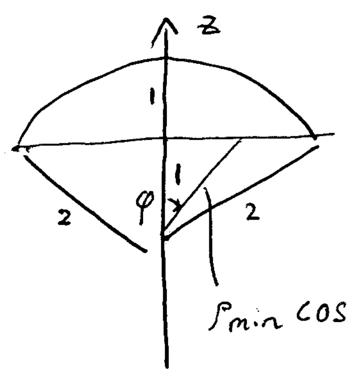


small cap above $z=1$ plane

large bowl beneath $z=1$ plane

(iii) The cap is smaller, its X-section looks like





$\rho_{min} \cos\varphi = 1$

$0 \leq \theta < 2\pi$

$0 \leq \varphi \leq \frac{\pi}{3}$

$\sec\varphi \leq \rho \leq 2$

Working for question 2

$$\text{Volume} \; = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{3}} \sin\varphi \, d\varphi \int_{\sec\varphi}^{2} \rho^2 \, d\rho$$

$$= 2\pi \int_0^{\frac{\pi}{3}} \sin\varphi \left[\tfrac{1}{3}\rho^3 \right]_{\sec\varphi}^{1} d\varphi$$

$$= \frac{2\pi}{3} \int_0^{\pi/3} \left[8\sin\varphi - \frac{\sin\varphi}{\cos^3\varphi} \right] d\varphi$$

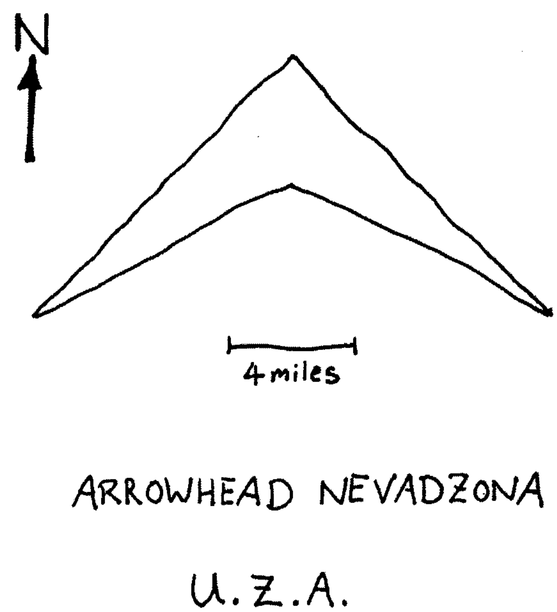
$$= \frac{2\pi}{3} \left[-8\cos\varphi \right]_0^{\pi/3} + \frac{2\pi}{3} \int_{\cos\varphi=1}^{\cos\varphi=\frac{1}{2}} \frac{d(\cos\varphi)}{\cos^3\varphi}$$

$$= \frac{2\pi}{3}\left[-\frac{8}{2} + 8 \right] + \frac{2\pi}{3} \left[-\frac{1}{2}\,\frac{1}{\cos^2\varphi} \right]_{\cos\varphi=1}^{\cos\varphi=\frac{1}{2}}$$

$$= \frac{8\pi}{3} + \frac{2\pi}{3}\left[-2 + \frac{1}{2} \right] = \frac{5\pi}{3}$$

Question 3 *30 points*

The town of Arrowhead is depicted in the map below:

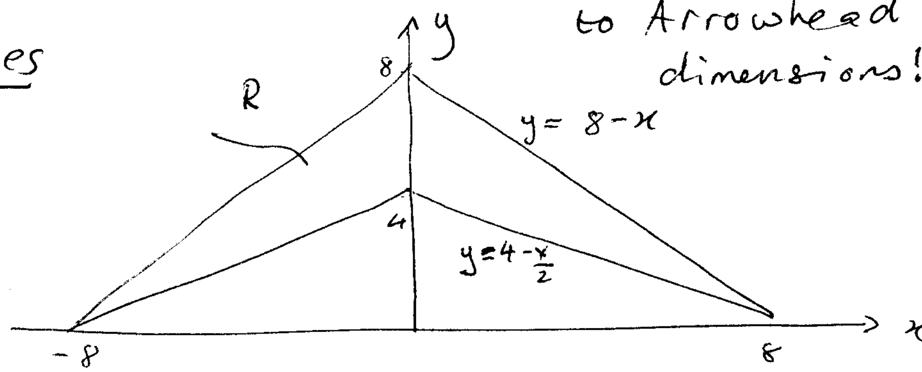


The population density rises steadily from zero in the deep South to a maximum of 1,000,000 people/mile^2 in the North. Use double integration to estimate (i) the total population and (ii) the average population density per square mile in Arrowhead.

[Accept any REASONABLE approximation to Arrowhead's dimensions!]

Working for question 3

Coordinates



$$\underset{2\Delta\text{'s}}{\text{AREA}} = 2 \times \frac{4 \times 8}{2} = 32 \text{ sq miles}$$

$$\text{POPULATION DENSITY} = \frac{1{,}000{,}000}{8}\, y = \underset{\frac{\text{people}}{\text{Mile}^2}}{\rho(x,y)}$$

$$\text{TOTAL POP.} = \int_R dA\, \rho = 2 \int_0^8 dx \int_{4-\frac{x}{2}}^{8-x} dy\, \frac{1000000}{8}\, y$$

by symmetry

$$= \frac{1000000}{4} \int_0^8 dx \left[+\tfrac{1}{2}(8-x)^2 - \tfrac{1}{2}\left(4-\tfrac{x}{2}\right)^2 \right]$$

$$= \frac{1000000}{4}\, \frac{1}{2} \left[-\tfrac{1}{3}(8-x)^3 + \tfrac{2}{3}\left(4-\tfrac{x}{2}\right)^3 \right]_0^8$$

$$= \frac{1000000}{4}\, \frac{1}{2} \cdot 128 = 16{,}000{,}000 \text{ people}$$

$$\text{AV. POP. DENSITY} = \frac{\text{TOTAL POP}}{\text{AREA}} = \frac{16\,000\,000}{32} = 500000 \ \frac{\text{people}}{\text{mile}^2}$$