240A DIFFERENTIAL GEOMETRY

Syllabus

- Differentiable Manifolds
  Vectors, Tensors, Forms, Lie Derivatives
- Symplectic Structures
  Classical Mechanics
- (pseudo) Riemannian Structures
  Metrics, Affine Connection, Curvature, Geodesics
- Conformal Structures

Grading Determined by Plebiscite

Recommended reading

Spivak 5 volumes A comprehensive introduction to DG

Choquet-Bruhat et al Analysis Manifolds & Physics

Eguchi, Gilkey, Hanson Phys. Rep. 66 213 1980

Hawking & Ellis Large Scale structure of Spacetime

Kobayashi Nomizu 2 volumes Foundations of DG

do Carmo Riemannian Geometry

Eisenhart (old) An Introduction to DG

DIFFERENTIABLE MANIFOLDS

Let M be a topological manifold:

i.e. M is a topological space

$[(M, \{V_i\}),\ \phi, M \in \{V_i\},\ \bigcup_\alpha V_\alpha \in \{V_i\},\ \bigcap_\beta^{\#\beta<\infty} V_\beta \in \{V_i\}]$

with properties

- $\forall\, P, Q \in M\ \ \exists\, U \ni P,\ V \ni Q$ open, disjoint, subsets of M (HAUSDORFF)
- The topology for M has a countable basis (SECOND COUNTABLE)
- $\forall\, P \in M,\ \exists\, U \subset M$ an open subset containing P and homeomorphic to a subset of $\mathbb{R}^n$ (LOCALLY n-DIMENSIONAL EUCLIDEAN)

The cartographer's approach:

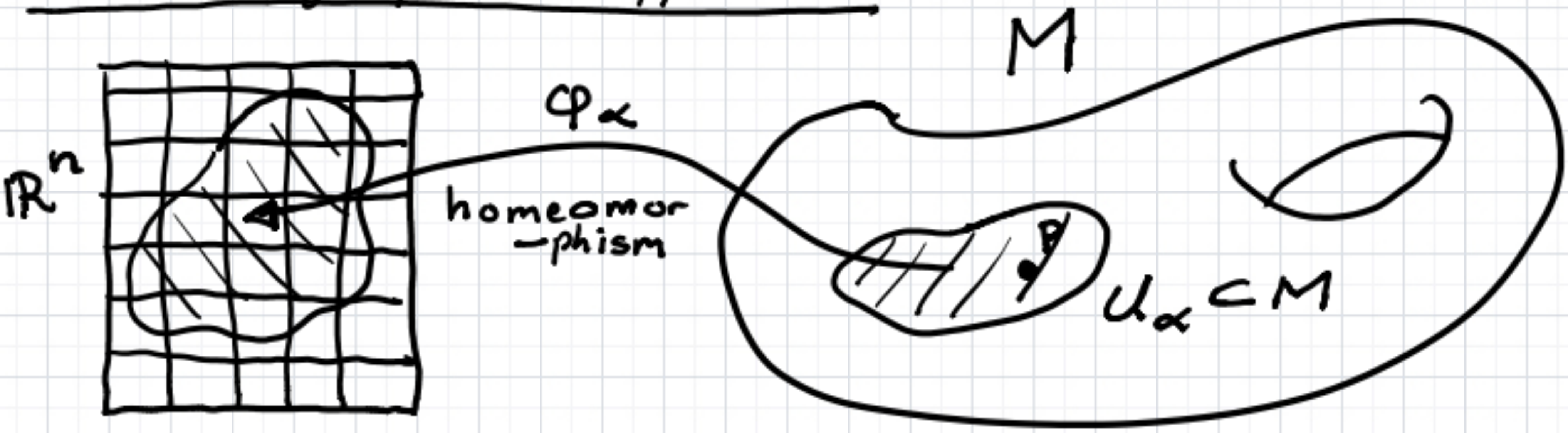
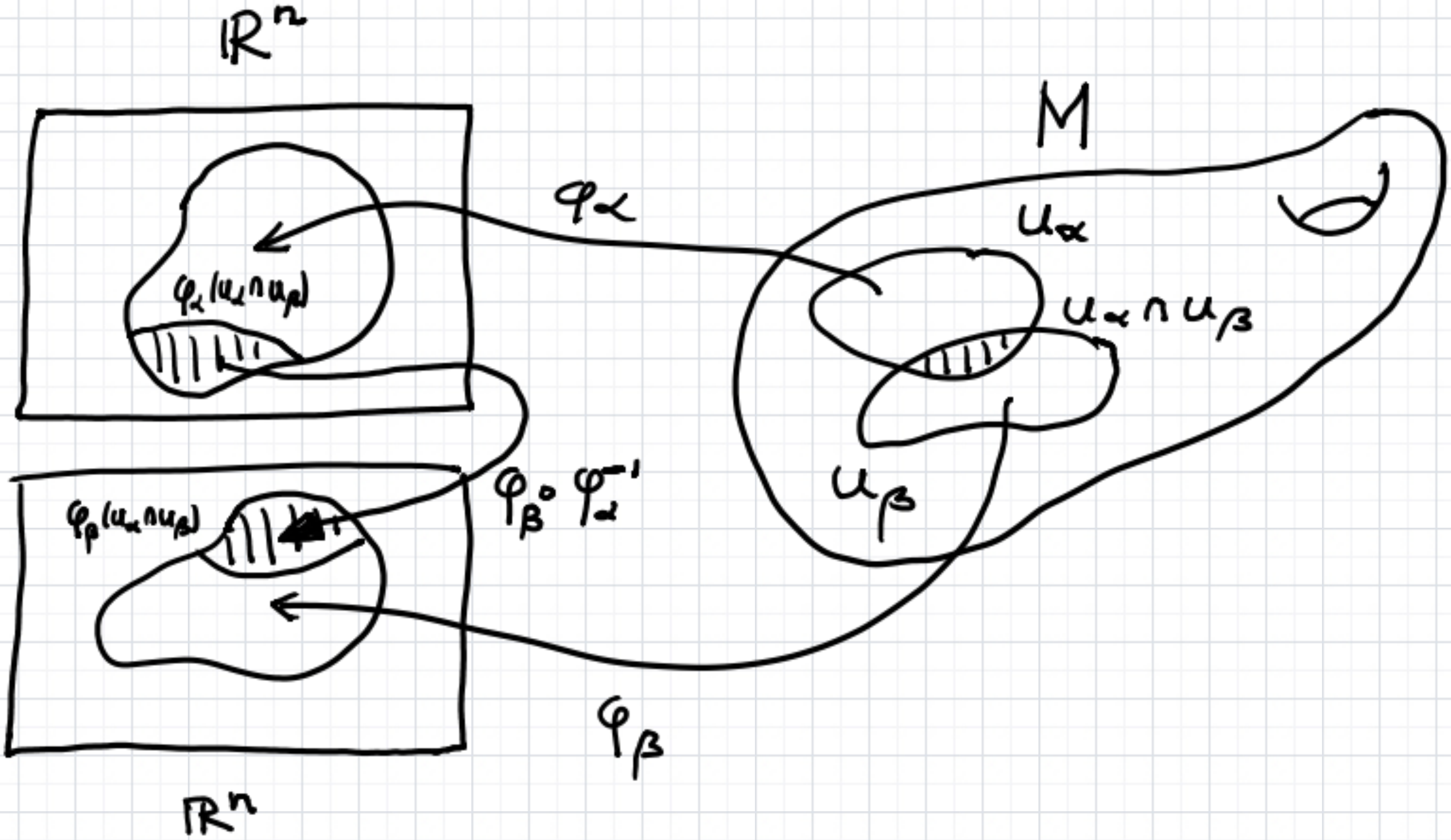


Fig.1 A CHART $(U_\alpha, \varphi_\alpha)$

Add structure to M by requiring compatiability of charts:



$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

HOMEOMORPHISM B/W OPEN SUBSETS OF $\mathbb{R}^n$

The space $\mathbb{R}^n$ has a lot of well understood structures:

- Differentiable
- C^k k-fold differentiable functions
- C^∞ smooth
- $\mathbb{R}^{2n} \cong \mathbb{C}^n \to$ holomorphic

REQUIRING "TRANSITION FUNCTIONS" $\varphi_\beta \circ \varphi_\alpha^{-1}$ to lie in the above classes yields

- DIFFERENTIABLE MANIFOLD
- C^k MANIFOLD
- SMOOTH MANIFOLD
- COMPLEX MANIFOLD

Their study is called "DIFFERENTIAL GEOMETRY" because these differentiable structures allow us apply calculus to manifolds.

We call charts $(U_\alpha, \varphi_\alpha)$ & (U_β, φ_β) <u>compatible</u> when $\varphi_\beta \circ \varphi_\alpha^{-1}$ is <u>differentiable</u>.

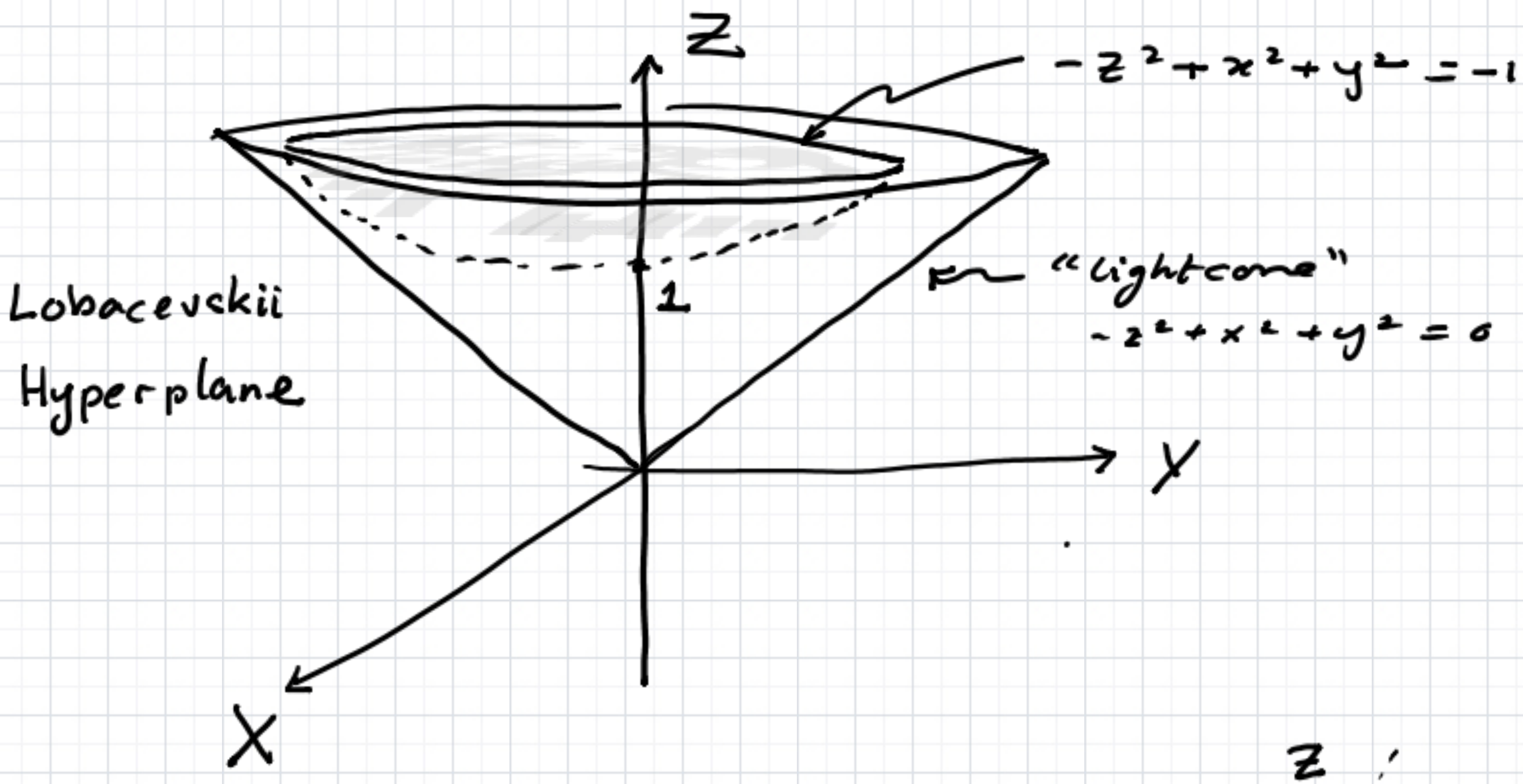
A set of compatible charts $\{(U_\alpha, \varphi_\alpha)\}$ covering M (i.e. $M = \bigcup_\alpha U_\alpha$) is called an <u>atlas</u> (or differentiable structure).

A topological space equipped with an altas is called a <u>differentiable manifold</u>.

<u>Note</u>: Strictly speaking, an atlas constitutes a single "coordinate system" for a space M. Taking the union over all compatible charts gives a <u>maximal atlas</u> which is usually taken as part of the definition of a differentiable manifold.

Examples

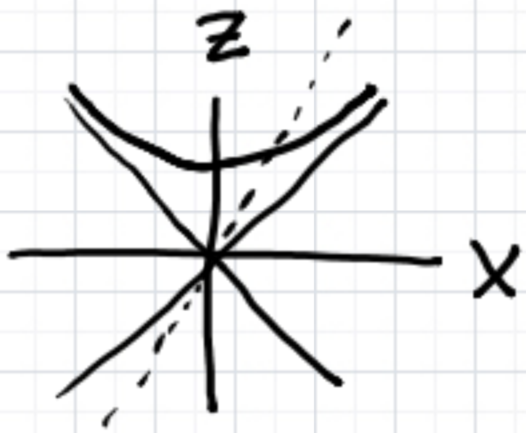
① $M = \{(X,Y,Z) : -Z^2+X^2+Y^2 = -1,\ Z>0\}$



Let $\varphi : M \longrightarrow U \subset \mathbb{R}^2$

$(X,Y,Z) \longmapsto \left(\frac{X}{Z}, \frac{Y}{Z}\right)$

Here U is the open unit disc.



use inverse slope as coordinates

This space is famous as the first non-Euclidean geometry:

Let $(x,y) \in \mathcal{U}$. We will often call these "coordinates" for M. Since there are two coordinates we call M 2-dimensional.

Notice $\varphi^{-1} : (x,y) \longmapsto \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, \frac{1}{\sqrt{1-x^2-y^2}} \right)$

and only a single chart is needed, so there is little to check.

Nonetheless, $M \equiv \mathbb{H}^2$ is a very important example of a manifold (also called Poincaré disc or hyperbolic plane or upper half plane).

Using the differential distance $ds^2 = +\underbrace{|dX + i\,dY|^2}_{\text{positive}} - \underbrace{dZ^2}_{\text{negative}}$

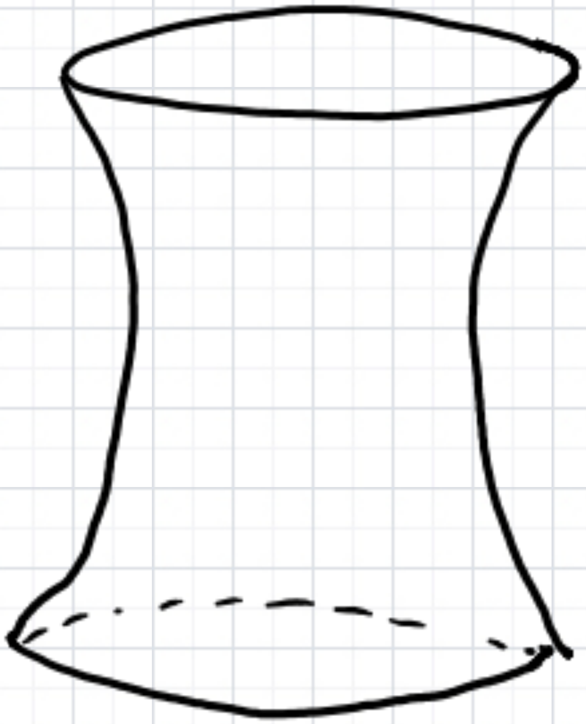
we can compute distances on the Poincaré disc $(z = x+iy)$

$$ds^2_{\mathbb{H}^2} = \frac{1}{4} \frac{|z\,d\bar{z}|^2}{(1-|z|^3)} \geq 0$$

"Euclidean geometry" of constant -ve curvature.

② $M = \{(Z, \vec{X}) : -Z^2 + \vec{X}^2 = +1\}$

i.e.



hyperboloid of a single sheet.

NEED TWO CHARTS

Equipped with the ambient distance

$$ds^2 = -dZ^2 + d\vec{X}^2$$

this is de Sitter space. For $\vec{X} \in \mathbb{R}^4$, it is a solution to Einstein's equations that approximates our Universe rather well! Positive, constant scalar curvature ("Cosmological Constant", Einstein's greatest mistake.

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Exercises

① $\mathbb{R}P_n$ = space of straight lines through origin in $\mathbb{R}^{n+1}$.

Define equivalence classes

$$[x^1, \ldots, x^{n+1}] = [\lambda x^1, \ldots, \lambda x^{n+1}] \qquad \lambda \neq 0$$

Use patch maps

omit $1 = \frac{x^i}{x^i}$

$$\varphi_i : [x^1, \ldots, x^{n+1}] \longmapsto \left(\frac{x^1}{x^i}, \ldots, \frac{x^{n+1}}{x^i} \right) \in \mathbb{R}^n$$

to show $\mathbb{R}P^n$ is a differentiable manifold.

② Let $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0 \right\} = Gl(2, \mathbb{R})$

$$P = \left\{ \begin{pmatrix} \alpha & \\ \beta & \gamma \end{pmatrix} : \gamma\alpha \neq 0 \right\} \subset G$$

a define an equivalence relation $g \sim g'$ $\quad g, g' \in G$

when $g = pg'$ for some $p \in P$.

Can you find a differentiable structure on $G/\sim$?