Lecture #10 DIFFERENTIAL FORMS

Recall, $\binom{m}{n}$-tensors mapped

$$\otimes^m T_p^* M \otimes \otimes^n T_p N \longrightarrow \mathbb{R}$$

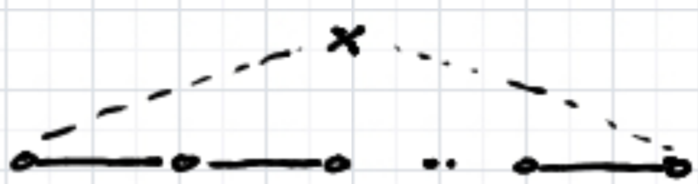
and were classified by $gl(\dim M, \mathbb{R})$ representations:

Note $\quad gl(n) \cong mat_n^n(\mathbb{R}) \quad$ and $\quad Gl(n)_+ = \exp gl(n)$

$\qquad\qquad$ = n×n matrices $\qquad\qquad$ = invertible n×n matrices

$$gl(n) = \mathbb{R} \oplus sl(n) \qquad\qquad Gl(n)_+ = \mathbb{R}^+ \otimes sl(n)$$

$sl(n)$ has Dynkin diagram



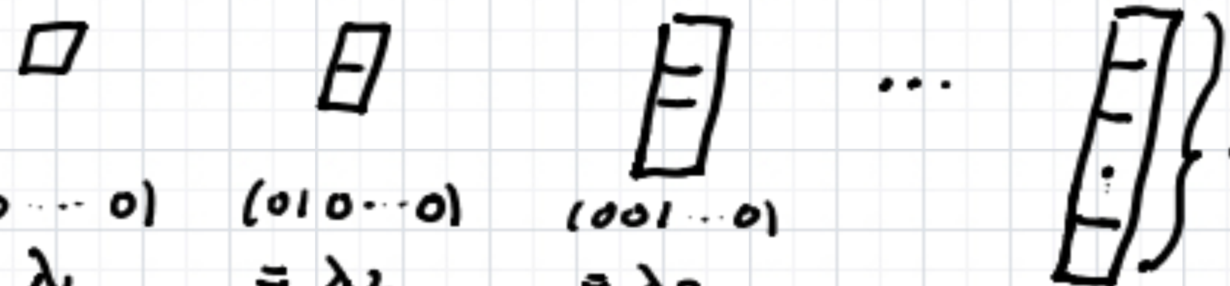
which gives information about <u>roots</u>.

<u>Roots</u> are vectors built from eigenvalues of a maximally commuting set of generators h_i acting on $\mathfrak{g}$ by commutation $[h_i, \cdot]$.

Irreducible representations are labeled by weights corresponding to sums of roots.

Concentrate on totally antisymmetric representions

$\emptyset$



n boxes

$(00\cdots0)$ $(10\cdots0) = \lambda_1$ $(010\cdots0) = \lambda_2$ $(001\cdots0) = \lambda_3$ $\cdots$ $(00\cdots1) = \lambda_{n-1}$

← sequence terminates in any given dimension

X x_i $x_{ij} = -x_{ji}$ x_{ijk} a/s $x_{i_1 i_2 \cdots i_n}$ a/s

$$\boxed{\dim R_{\lambda_k} = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}}$$

1 n $\frac{n(n-1)}{2}$ $\cdots$ $\frac{n(n-1)}{2}$ n 1

0-form function   1-form   2-form   vector   top form, volume

Recall that we could talk of differentiable tensors by giving the tensor bundle TM a differentiable structure inherited from M. The same applies specializing to antisymmetric tensors and yields <u>differential forms</u>.

Exterior Algebra

We call $\omega \in TM$ an <u>exterior differential p-form</u> if ω is of tensor type $\binom{0}{p}$ <u>and</u> <u>totally antisymmetric</u>

$$\omega(\ldots, v, \ldots, u \ldots) = -\omega(\ldots, u, \ldots, v \ldots) \in \mathbb{R}$$

for every pair of slots & $u, v \in \mathfrak{X}M$

We call differentiable functions $f: M \longrightarrow \mathbb{R}$ <u>0-forms</u>. The space ΛM of all p-forms ($0 \leq p \leq n = \dim M$) is called the exterior algebra bundle. It is endowed $\bar{c}$ a <u>product</u>:

Exterior / Wedge / Grassmann product

Call $\Lambda^p M$ the space of p-forms, then

$$\wedge : \Lambda^p M \times \Lambda^q M \longrightarrow \Lambda^{p+q} M$$
$$(\alpha, \beta) \longmapsto \alpha \wedge \beta$$

where the $\binom{0}{p+q}$ tensor $\alpha \wedge \beta$ acts on $p+q$ vectors $(v_1 \ldots v_{p+q})$ via

$$\boxed{(\alpha \wedge \beta)(v_1, \ldots, v_{p+q}) = \frac{1}{(p+q)!} \sum_{\sigma \in S(p+q)} \operatorname{sign}(\sigma) \cdot \sigma\left[\alpha(v_1 \ldots v_p)\, \beta(v_{p+1}, \ldots, v_{p+q})\right]}$$

Example Let θ^i be a basis for T^*U for some $U \subset M$ so that

$$\alpha = \alpha_{i_1 \cdots i_p} \theta^{i_1} \otimes \cdots \otimes \theta^{i_p}$$

$$\beta = \beta_{i_1 \cdots i_q} \theta^{i_1} \otimes \cdots \otimes \theta^{i_q}$$

then

$$\alpha \wedge \beta = \alpha_{i_1 \cdots i_p} \beta_{i_{p+1} \cdots} \beta_{i_{p+q}} \theta^{i_1} \wedge \cdots \wedge \theta^{i_{p+q}}$$

where

$$\boxed{\theta^{i_1} \wedge \cdots \wedge \theta^{i_k} = \frac{1}{k!} \sum_{\sigma \in S(k)} \operatorname{sgn}(\sigma)\, \theta^{i_{\sigma(1)}} \otimes \cdots \otimes \theta^{i_{\sigma(k)}}}$$

Hence for 1-forms α, β, γ.

$$\alpha \wedge \beta = \frac{1}{2} (\alpha \otimes \beta - \beta \otimes \alpha)$$

$$\alpha \wedge \beta \wedge \gamma = \frac{1}{3!} (\alpha \otimes \beta \otimes \gamma \pm 5 \text{ more})$$

For α a p-form & β a q-form

$$\boxed{\alpha \wedge \beta = (-)^{pq} \beta \wedge \alpha} \quad (*)$$

Moreover the exterior product is bilinear and associative. (*) says it is <u>graded</u> commutative.

The exterior derivative

$$d : \Lambda^p M \longrightarrow \Lambda^{p+1} M \quad \text{———} (*)$$

is defined in a coordinate system as

$$\boxed{d\left(\alpha_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}\right) = \left(\partial_i \alpha_{i_1 \ldots i_p}\right) dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}}$$

i.e.

$$d = dx^i \frac{\partial}{\partial x^i} \wedge .$$

It obeys

(i) Linear $\quad d(\lambda\alpha + \mu\beta) = \lambda d\alpha + \mu d\beta$

(ii) Graded Leibnitz

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-)^{p_\alpha} \alpha \wedge \beta$$

(iii) $d^2 = 0$

(iv) if $f : M \to \mathbb{R}$, $\quad df$ is the differential of f.

Notice $(*)$ holds because even though

$$dx^i \partial_i X$$

is not a tensor its totally antisymmetric part is.

Example $d=3$

function $\quad f \xrightarrow{d} \frac{\partial f}{\partial x^i} dx^i$ **gradient**

1-form $\quad \alpha_x dx + \alpha_y dy + \alpha_z dz \xmapsto{d} (\partial_x \alpha_y - \partial_y \alpha_x)\, dx \wedge dy$

$$+ (\partial_y \alpha_z - \partial_z \alpha_y)\, dy \wedge dz \quad \textbf{curl}$$

$$+ (\partial_z \alpha_x - \partial_x \alpha_z)\, dz \wedge dx$$

2-form $\quad \beta^x dy \wedge dz + \beta^y dz \wedge dx + \beta^z dx \wedge dy \xmapsto{d} (\partial_x \beta^x + \partial_y \beta^y + \partial_z \beta^z)$

$$\cdot\, dx \wedge dy \wedge dz$$

divergence

3-form $\quad \gamma\, dx \wedge dy \wedge dz \xmapsto{d} 0$

Then $d^2 = 0$ summarizes $\quad$ curl grad $= 0$

div curl $= 0$

Note If $\alpha = d\beta$ for some β we call α exact

If $d\alpha = 0$ we call α closed

The operator d defines the de Rham complex

$$0 \longrightarrow \Lambda^0 M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^{n-1} M \xrightarrow{d} \Lambda M \longrightarrow 0$$

Its cohomology $\boxed{\dfrac{\text{closed forms}}{\text{exact forms}} = \text{de Rham cohomology}}$

Next time, classical mechanics

Exercise: ① Suppose the p-form α is closed on a region $U \subset M$ such that U is diffeomorphic to an open ball in $\mathbb{R}^n$.

Consider

$$\beta_{i_1 \ldots i_{p-1}} = \int_0^1 t^{p-1} x^i \alpha_{i i_1 \ldots i_{p-1}}(tx^1, \ldots, tx^n)\, dt$$

Compute $d\beta$. Comment on your conclusion.

② Find out what the "Poincaré Lemma" says about closed forms.