Lecture # 11 EXTERIOR DERIVATIVE

$$d : \Lambda^p M \longrightarrow \Lambda^{p+1} M \qquad \text{——} (*)$$

is defined in a coordinate system as

$$\boxed{d\left(\alpha_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}\right) = \left(\partial_i \alpha_{i_1 \dots i_p}\right) dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}}$$

i.e.

$$d = dx^i \frac{\partial}{\partial x^i} \wedge .$$

It obeys

(i) Linear $\quad d(\lambda\alpha + \mu\beta) = \lambda d\alpha + \mu\, d\beta$

(ii) Graded Leibnitz

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-)^{p_\alpha} \alpha \wedge \beta$$

(iii) $d^2 = 0$

(iv) if $f : M \to \mathbb{R}$, $\; df$ is the differential of f.

Notice $(*)$ holds because even though

$$dx^i \partial_i X$$

is not a tensor its totally antisymmetric part is.

Example $d=3$

function $f \xrightarrow{d} \frac{\partial f}{\partial x^i} dx^i$ **gradient**

1-form $\alpha_x dx + \alpha_y dy + \alpha_z dz \xrightarrow{d} (\partial_x \alpha_y - \partial_y \alpha_x)\, dx \wedge dy$

$+ (\partial_y \alpha_z - \partial_z \alpha_y)\, dy \wedge dz$ **curl**

$+ (\partial_z \alpha_x - \partial_x \alpha_z)\, dz \wedge dx$

2-form $\beta^x dy \wedge dz + \beta^y dz \wedge dx + \beta^z dx \wedge dy \xmapsto{d} (\partial_x \beta^x + \partial_y \beta^y + \partial_z \beta^z)$

$\cdot\, dx \wedge dy \wedge dz$

divergence

3-form $\gamma\, dx \wedge dy \wedge dz \xmapsto{d} 0$

Then $d^2 = 0$ summarizes curl grad $= 0$

div curl $= 0$]

Note If $\alpha = d\beta$ for some β we call α **exact**

If $d\alpha = 0$ we call α **closed**

The operator d defines the de Rham complex

$$0 \longrightarrow \Lambda^0 M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^{n-1} M \xrightarrow{d} \Lambda M \longrightarrow 0$$

Its **cohomology** $\boxed{\frac{\text{closed forms}}{\text{exact forms}} = \text{de Rham cohomology}}$

Classical mechanics

Symplectic Manifold

We call Z a symplectic manifold if it has a closed, non-degenerate 2-form ω. [Assume Z, ω differentiable]

Non degenerate means $\omega = \omega_{AB} \, dz^A \wedge dz^B$ obeys $\boxed{\det \omega_{AB} \neq 0.}$ $\Rightarrow$ dim Z is even

Poincaré Lemma

On any contractible domain, a closed p-form α can be written

$$\alpha = d\beta$$

for some p- form β.

* HW gives a idea for a proof
* $d\alpha = 0$ follows for free since $d^2\beta = 0$
* β is not unique, $\beta \to \beta + d\gamma$ is still a solution. "Gauge symmetry". $\gamma \to \gamma + d\delta$ is called a "gauge for gauge" symmetry.

<u>Example</u> $T^*M = \{(p,\theta_p) : p \in M,\ \theta_p \in T_p^*M\}$

The cotangent bundle is itself a differentiable manifold. At every point we assign

$$(p,\theta_p) \longmapsto \theta_p$$

"Tautological one form".

In a coordinate system

$$\left(x^\mu,\ \theta_p = p_\mu\, dx^\mu_p\right) \longmapsto p_\mu\, dx^\mu$$

$$\parallel$$

$$Z^A = (x^\mu, p_\mu)$$

If we view this as a symplectic current

$$\theta = p_\mu\, dx^\mu$$

then

$$\omega = d\theta = d(p_\mu\, dx^\mu) = dp_\mu \wedge dx^\mu$$

Here use x^μ (rather than x^i since may want to interpret one coordinate as a time).

Since $dp_\mu \wedge dx^\mu = \frac{1}{2} dp_\mu \wedge dx^\mu - \frac{1}{2} dx^\mu \wedge dp_\mu$

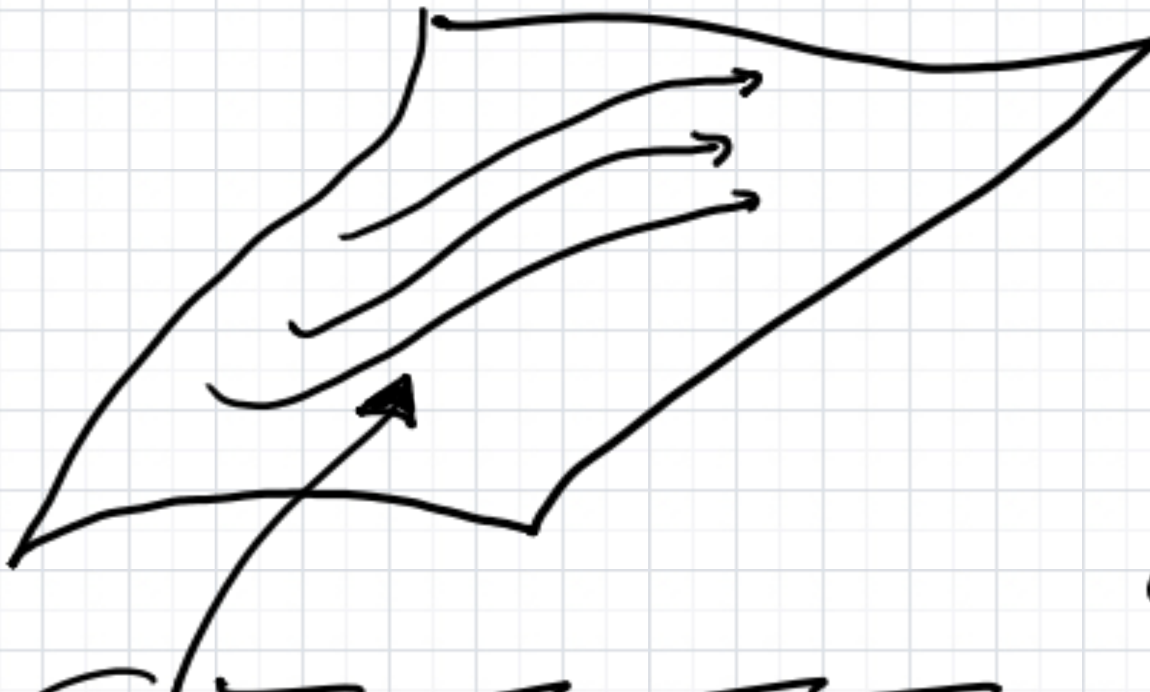
$$= \omega_{AB}\, dz^A \wedge dz^B$$

we have

$$\omega_{AB} = \left(\begin{array}{c|c} & \frac{1}{2}\delta_\mu^\nu \\ \hline -\frac{1}{2}\delta_\nu^\mu & \end{array} \right) \quad \underline{\text{invertible}}$$

Actually, a basic theorem of symplectic geometry is that there always exists a coordinate chart around $p \in Z$ such that $\omega = dx_i \wedge dy^i$ $[\varphi_\alpha : p \mapsto (x_i, y^i)]$ called <u>Darboux theorem</u>. "Symplectic geometry is more rigid than Riemannian geometry".

Hamiltonian dynamics



Z = Symplectic manifold

"Phase space"

or "generalized positions & momenta"

Want to find integral curves or flows that are consistent with positions and momenta

Suppose $H : Z \longrightarrow \mathbb{R}$ is some function.

Then dH is a 1-form so if we had a $\binom{2}{0}$-tensor Ω (say)

$$\Omega(dH, \cdot) \in \mathfrak{X} Z$$

because its second slot maps 1-forms to real numbers.

But on a symplectic manifold, there is a canonical choice for Ω:

$$\Omega = (\omega_{AB})^{-1} \frac{\partial}{\partial z^A} \otimes \frac{\partial}{\partial z^B}$$

$$\equiv \Omega^{AB}$$

because ω is non-degenerate.

Ω is called the Poisson bivector.

The vector field

$$X_H = \Omega(dH, \cdot)$$

is called a Hamiltonian vector field.

Let $f : Z \longrightarrow \mathbb{R}$. The job of a vector field is to compute the directional derivative of functions along integral curves to the field.

I.e. From the gradient of f, form a 1-form df on which X_H can act.

In equations

$$\boxed{\dot{f} = \Omega(dH, df)}$$

$\frac{d}{dt}$ along flows

These are called Hamilton's equations.

Usually they are written slightly differently by introducing the "Poisson bracket"

$$\{\cdot,\cdot\}_{PB} = \Omega(d\cdot, d\cdot)$$

$$\Rightarrow \quad \dot{f} = \{H, f\}_{PB} \quad \text{or} \quad \frac{d}{dt} = \{H, \cdot\}_{PB}$$

If you are familiar with quantum mechanics, the second equation should remind you of the Schrödinger equation.

Review Exercise: Find an example of a closed but non-exact differential form.