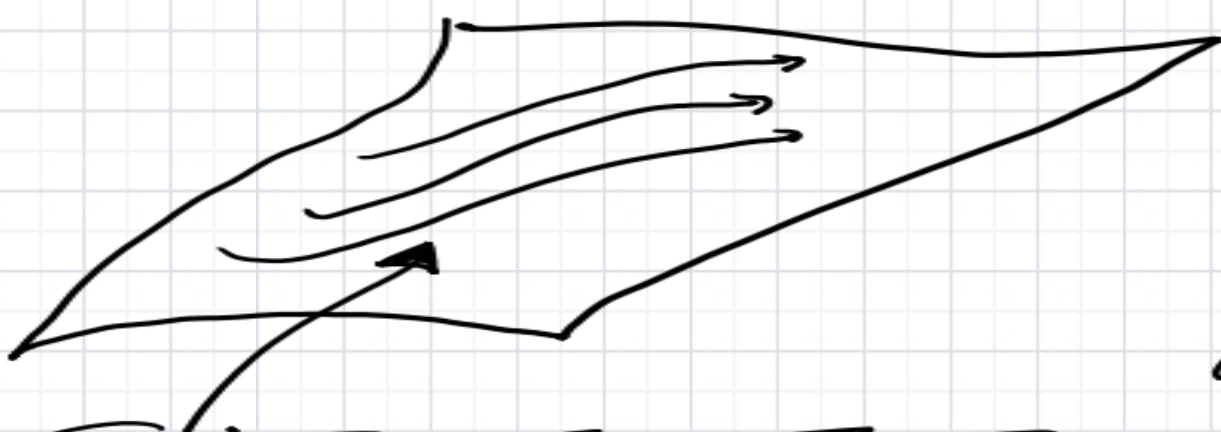
Lecture #12 CLASSICAL MECHANICS

* Symplectic Manifold (Z, ω) is an even dimensional manifold with a non-degenerate closed 2-form ω.

* $T^*M \supset U \xrightarrow[\varphi]{} (p_\mu, x^\mu)$ and $\omega = dp_\mu \wedge dx^\mu$

Hamiltonian dynamics



Z = Symplectic manifold

"Phase space" or "generalized positions & momenta"

Want to find integral curves or flows that are consistent with positions and momenta

Suppose $H : Z \longrightarrow \mathbb{R}$ is some function. Then dH is a 1-form so if we had a $\binom{2}{0}$-tensor Ω (say)

$$\Omega(dH, \cdot) \in \mathfrak{X} Z$$

because its second slot maps 1-forms to real numbers.

But on a symplectic manifold, there is a canonical choice for Ω:

$$\Omega = (\omega_{AB})^{-1} \frac{\partial}{\partial z^A} \otimes \frac{\partial}{\partial z^B}$$

$$\equiv \Omega^{AB}$$

because ω is non-degenerate.

Ω is called the Poisson bivector.

The vector field

$$X_H = \Omega(dH, \cdot)$$

is called a Hamiltonian vector field.

Let $f : Z \longrightarrow \mathbb{R}$. The job of a vector field is to compute the directional derivative of functions along integral curves to the field.

i.e. From the gradient of f, form a 1-form df on which X_H can act.

In equations

$$\dot{f} = \boxed{\Omega(dH, df)}$$

$\frac{d}{dt}$ along flows

These are called __Hamilton's equations__.

Usually they are written slightly differently by introducing the "Poisson bracket"

$$\{\cdot, \cdot\}_{PB} = \Omega(d\cdot, d\cdot)$$

$$\Rightarrow \quad \dot{f} = \{H, f\}_{PB} \quad \text{or} \quad \frac{d}{dt} = \{H, \cdot\}_{PB}$$

If you are familiar with quantum mechanics, the second equation should remind you of the __Schrödinger equation__.

Properties of Poisson Bracket

In Darboux coordinates

$$\Omega = \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial y_i} - \frac{\partial}{\partial y^i} \otimes \frac{\partial}{\partial x^i}$$

and

$$\{f, g\}_{PB} = \Omega(df, dg)$$

$$= \Omega\left(dx^i \frac{\partial f}{\partial x^i} + dy_i \frac{\partial f}{\partial y_i}, \; dx^j \frac{\partial g}{\partial x^j} + dy_j \frac{\partial g}{\partial x^j}\right)$$

$$= \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial y_i}$$

which is the formula you will find in Goldstein.

* $\{f, g\}_{PB} = - \{g, f\}_{PB}$
* $\{f, \{g, h\}_{PB}\}_{PB} + \text{cyclic} = 0$
* $\{f, gh\}_{PB} = \{f, g\}_{PB}\, h + g \{f, h\}_{PB}$

All follow from the definition.

Notice that $\{\cdot,\cdot\}_{PB}$ obeys properties similar to the Lie bracket. In fact it makes the space of differentiable functions on Z into a Lie algebra.

<u>Remark</u> Poisson Brackets are the basis for <u>quantization</u>

$$\{\cdot,\cdot\}_{PB} \longrightarrow \frac{1}{i\hbar}[\cdot,\cdot]$$

where functions on Z are replaced by operators acting on a Hilbert space.

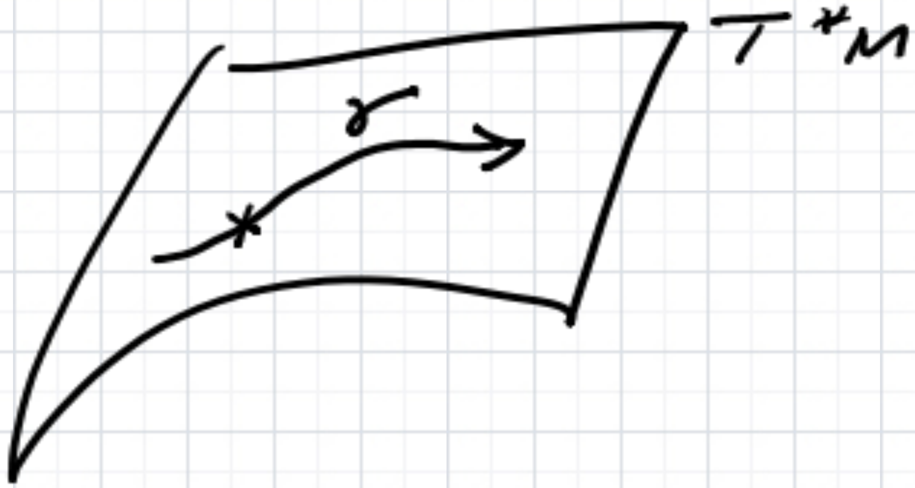
<u>Example</u> T^*M with $\omega = dp_\mu \wedge dx^\mu$

and $H: T^*M \longrightarrow \mathbb{R}$

Then $\Omega = \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial p^\mu} - \frac{\partial}{\partial p^\mu} \otimes \frac{\partial}{\partial x^\mu}$ [check coeff!]

Recall $\frac{d}{dt} = \{H,\cdot\}_{PB}$

<u>So</u> we can compute integral curves in T^*M



by calculating $\dot{\gamma} = (\dot{x}^\mu, \dot{p}_\mu)$

$$\begin{cases} \dot{x}^\mu = \{H, x^\mu\}_{PB} = \dfrac{\partial H}{\partial p_\nu}\dfrac{\partial x^\mu}{\partial x^\nu} - \dfrac{\partial H}{\partial x_\nu}\dfrac{\partial x^\nu}{\partial \nu} = \dfrac{\partial H}{\partial p_\mu} \\ \\ \dot{p}_\mu = -\dfrac{\partial H}{\partial x^\mu} \end{cases}$$ are Hamilton's equations.

<u>Next</u> choose an interesting H.

Suppose M has a $\binom{2}{0}$ tensor $g^{\mu\nu}\dfrac{\partial}{\partial x^\mu}\otimes\dfrac{\partial}{\partial x^\nu} = g$

Take $2H = g(\theta, \theta) = g^{\mu\nu}(x)\, p_\mu p_\nu$

Then

$$\begin{cases} \dot{x}^\mu = \dfrac{\partial}{\partial p_\mu}\left(\frac{1}{2} p_\rho g^{\rho\sigma} p_\sigma\right) \\ \dot{p}_\mu = -\dfrac{\partial}{\partial x^\mu}\left(\frac{1}{2} p_\rho g^{\rho\sigma} p_\sigma\right) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}^\mu = g^{\rho\sigma} p_\sigma \equiv p^\rho \\ \dot{p}_\mu = -\left(\dfrac{\partial}{\partial x^\mu} g^{\rho\sigma}\right) p_\sigma \end{cases}$$

Review Exercise

(i) Compute $\ddot{x}^\mu$

(ii) Look up the "geodesic" equation and compare your result to what you find.