Lecture # 13 GEODESICS

Last time we studied Hamilton's equations on T^*M with

$$H = \frac{1}{2} p_\mu g^{\mu\nu} p_\nu$$

Poisson bivector

and found that $\frac{d}{dt} = \{H, \cdot\}_{PB} = \omega^{-1}(dH, d\cdot)$

implied
$$\begin{cases} \dot{x}^\mu = g^{\mu\nu} p_\nu & \text{--- (1)} \\ \dot{p}_\mu = -\frac{1}{2} \partial_\mu g^{\rho\sigma} p_\rho p_\sigma & \text{--- (2)} \end{cases}$$

Assume $g^{\mu\nu}$ invertible with inverse $g_{\mu\nu}$

$$g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho$$

Then (1) says $p_\mu = g_{\mu\nu} \dot{x}^\nu$, plugging into (2) yields

$$\frac{d}{dt}(g_{\mu\nu} \dot{x}^\nu) + \frac{1}{2} \partial_\mu g^{\rho\sigma} g_{\rho\nu} \dot{x}^\nu g_{\sigma\kappa} \dot{x}^\kappa = 0$$

$$\Rightarrow \quad g_{\mu\nu}\ddot{x}^\nu + \dot{x}^\rho \partial_\rho g_{\mu\nu}\dot{x}^\nu + \tfrac{1}{2}\partial_\mu g^{\rho\sigma} g_{\rho\sigma}\dot{x}^\nu g_{\sigma\kappa}\dot{x}^\kappa = 0$$

But $\dot{M}^{-1} = -M^{-1}\dot{M}M^{-1}$ for matrices so

$$g_{\mu\nu}\ddot{x}^\nu + \dot{x}^\rho\dot{x}^\nu \partial_\rho g_{\mu\nu} - \tfrac{1}{2}\dot{x}^\rho\dot{x}^\sigma \partial_\mu g_{\rho\sigma} = 0$$

$$\Rightarrow \quad \ddot{x}^\mu + \dot{x}^\rho\dot{x}^\sigma \left[g^{\mu\nu}\partial_\rho g_{\sigma\nu} - \tfrac{1}{2} g^{\mu\nu}\partial_\nu g_{\rho\sigma} \right] = 0$$

Defining

$$\boxed{\Gamma^\mu_{\rho\sigma} = \tfrac{1}{2} g^{\mu\nu}\left[\partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma}\right]}$$

"Christoffel Symbols"

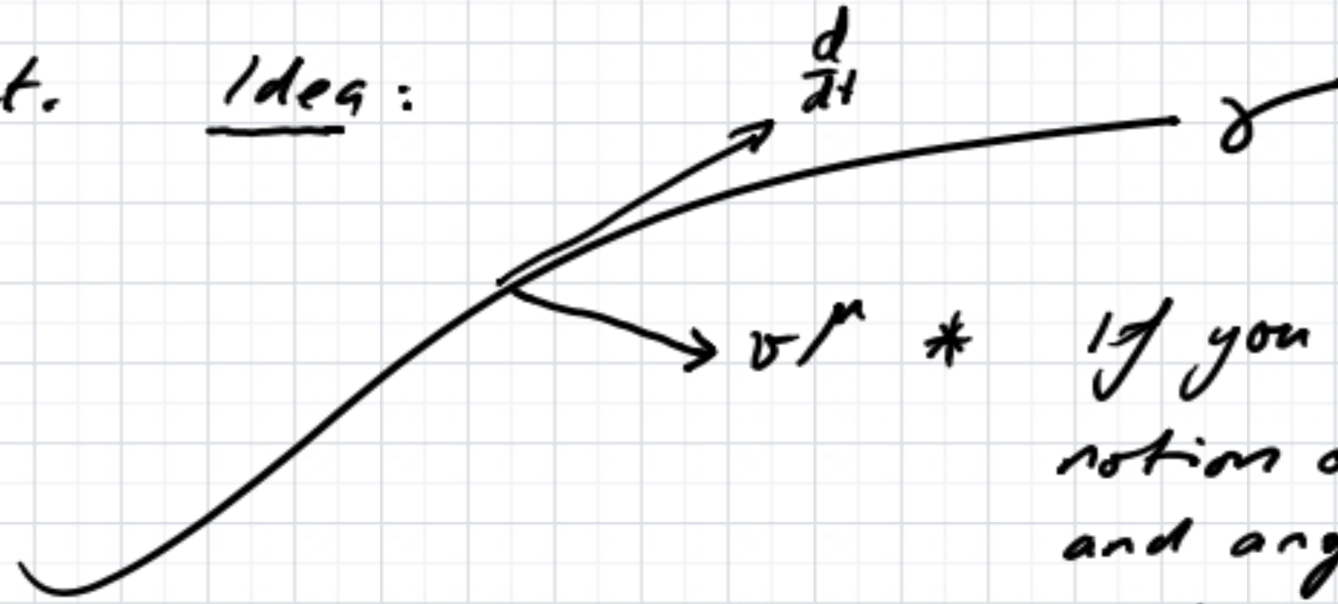
$$\Rightarrow \quad \frac{d}{dt}\dot{x}^\mu + \dot{x}^\rho \Gamma^\mu_{\rho\sigma}\dot{x}^\sigma = 0$$

For any vector v^μ call

$$\boxed{\frac{\nabla v^\mu}{dt} \equiv \dot{v}^\mu + \dot{x}^\rho \Gamma^\mu_{\rho\sigma} v^\sigma}$$

The equation $\frac{\nabla v^\mu}{dt} = 0$ is called parallel transport.

<u>Idea</u>:

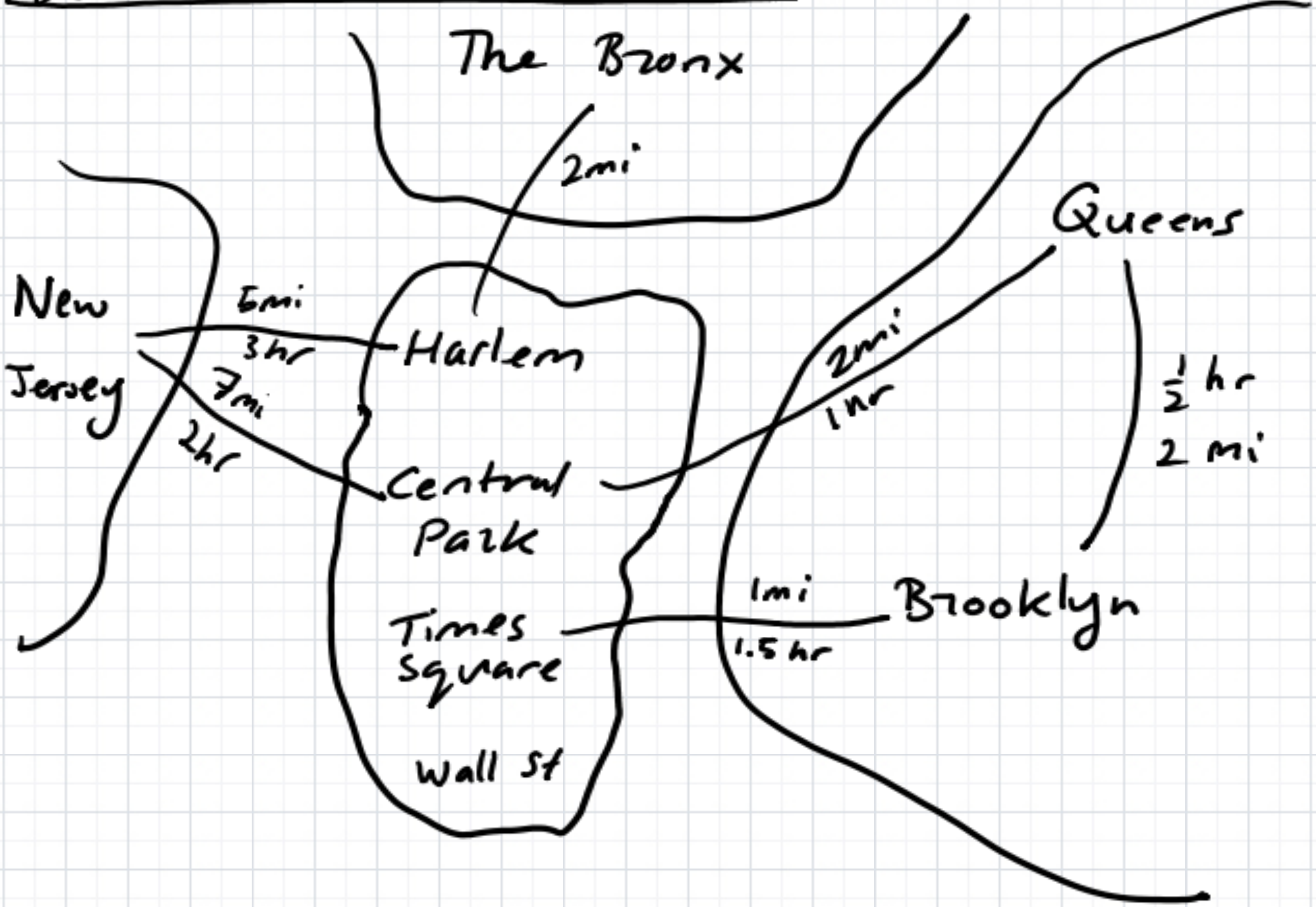


* If you have a notion of lengths and angles between vectors, it corresponds to holding the length and direction of v^μ constant along the path γ.

The requirement $\frac{\nabla \dot{x}^\mu}{dt} = 0$ means that the tangent vector $\dot{x}^\mu$ to γ has a constant <u>length</u> and <u>direction</u>.

Hence γ generalizes a straight line and is called a geodesic (notice that holding the <u>length</u> constant is not necessary here.)

Riemannian Geometry



We often want to measure distances on a manifold.

For a given manifold there a many ways to do this

A table listing distances b/w any pair of points would be cumbersome → want <u>local</u> definition.

__Idea__: Introduce a metric at every point $p \in M$ that measures lengths and angles b/w tangent vectors.

Compute distances along paths γ by integrating the length of the tangent vector $\dot{\gamma}$ along the path.

__Definition__: A __Riemannian manifold__ (M, g) is a differentiable manifold equipped with a $\binom{0}{2}$ tensor field g, called the __metric tensor__, which satisfies

(i) g is symmetric:

$$\forall\, X, Y \in \mathfrak{X}M, \quad g(X,Y) = g(Y,X)$$

(ii) g is differentiable

(iii) g is __non-degenerate__:

$\forall\, p \in M$

$$g_p : \otimes^2 T_pM \longrightarrow \mathbb{R} \text{ satisfies } g(X_p, Y_p) = 0$$

$\forall\, X_p \in T_pM$ iff $Y_p = 0$.

We call a Riemannian manifold (a Riemannian structure) <u>proper</u> if g is also <u>positive definite</u>.

i.e. $g_p(X_p, X_p) > 0 \quad \forall \; X_p \in T_pM, \; p \in M, \; X_p \neq 0$

Otherwise we say (M,g) is <u>pseudo</u>/<u>semi</u> Riemannian.

The tensor g is often called the <u>line element</u> and in coordinates written

$$g = ds^2 = g_{ij} \, dx^i \otimes dx^j \qquad g_{ij} = g_{ji}$$

$$= g_{ij} \, dx^i \odot dx^j$$

$$= g_{ij} \, dx^i dx^j$$

Some famous metrics

$\mathbb{R}^2$ $\quad ds^2 = dx^2 + dy^2$ standard metric on the plane

$\mathbb{H}$ $\quad ds^2 = \frac{dx^2+dy^2}{y^2}$ Lobachevskii's non-Euclidean geometry on upper $\frac{1}{2}$ plane $y > 0$ (sometimes called Poincaré disc or 2-dimensional hyperbolic space).

dS_4 $\quad ds^2 = -dt^2 + e^{2Mt} d\vec{x}^2$ de Sitter space (our universe).

S^2 $\quad d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$ standard metric on the sphere

$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$ Schwarzschild black hole

$\mathbb{R}^{3,1}$ $\quad d\sigma^2 = -dt^2 + dx^2 + dy^2 + dz^2$ Flat "Minkowski" spacetime.

The metric tensor g gives T_pM an inner product

$$(\cdot,\cdot) : \odot^2 T_pM \longrightarrow \mathbb{R}$$

via

$$(v,w) = g_p(v,w), \text{ when } (M,g) \text{ is } \underline{\text{proper}}.$$

In turn $\|v\| = \sqrt{g_p(v,v)}$ is a norm.

Since $g_{ij} = g_p(e_i, e_j)$, the components of g in some basis $\{e_i\}$ for T_pM, is a symmetric, non degenerate, positive matrix the norm & inner product properties follow trivially.

In fact, <u>at a point p</u>, we can always find a basis for any metric g where

$$(g_{ij}) = \operatorname{diag}(\pm 1, \pm 1, \dots, \pm 1).$$

The *signature* of g is the number of plus and minus signs, for example, $\mathbb{R}^{3,1}$ is $(-,+,+,+)$.

For the pseudo-Riemannian case, can still define $\|v\|^2 = g_p(v,v)$ and call $v \in T_pM$

(i) timelike $\|v\|^2 < 0$ "real particles"

(ii) null/lightlike/isotropic $\|v\|^2 = 0$ "light"

(iii) spacelike $\|v\|^2 < 0$ "tachyons"