Lecture # 14 Riemannian Geometry

$$g = ds^2 = (\cdot\,,\cdot) : \odot^2 TM \longrightarrow \mathbb{R}$$ (non-degenerate, differentiable)

gives T_pM an inner product and norm (when g is proper).

At any $p \in M$ we can diagonalize $g_{ij} = (e_i, e_j)$ and by $gl(\dim M)$ transformations achieve

$$g_{ij} = \operatorname{diag}(\pm 1, \pm 1, \ldots, \pm 1)$$

The signature of g is the number of plus or minus signs (it is an invariant because g is non-degenerate and gl transformations are densified by the sign of their determinant).

For example $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ is signature $(-+++)$.

Pseudo Riemannian structures are endowed with <u>causal structures</u>.

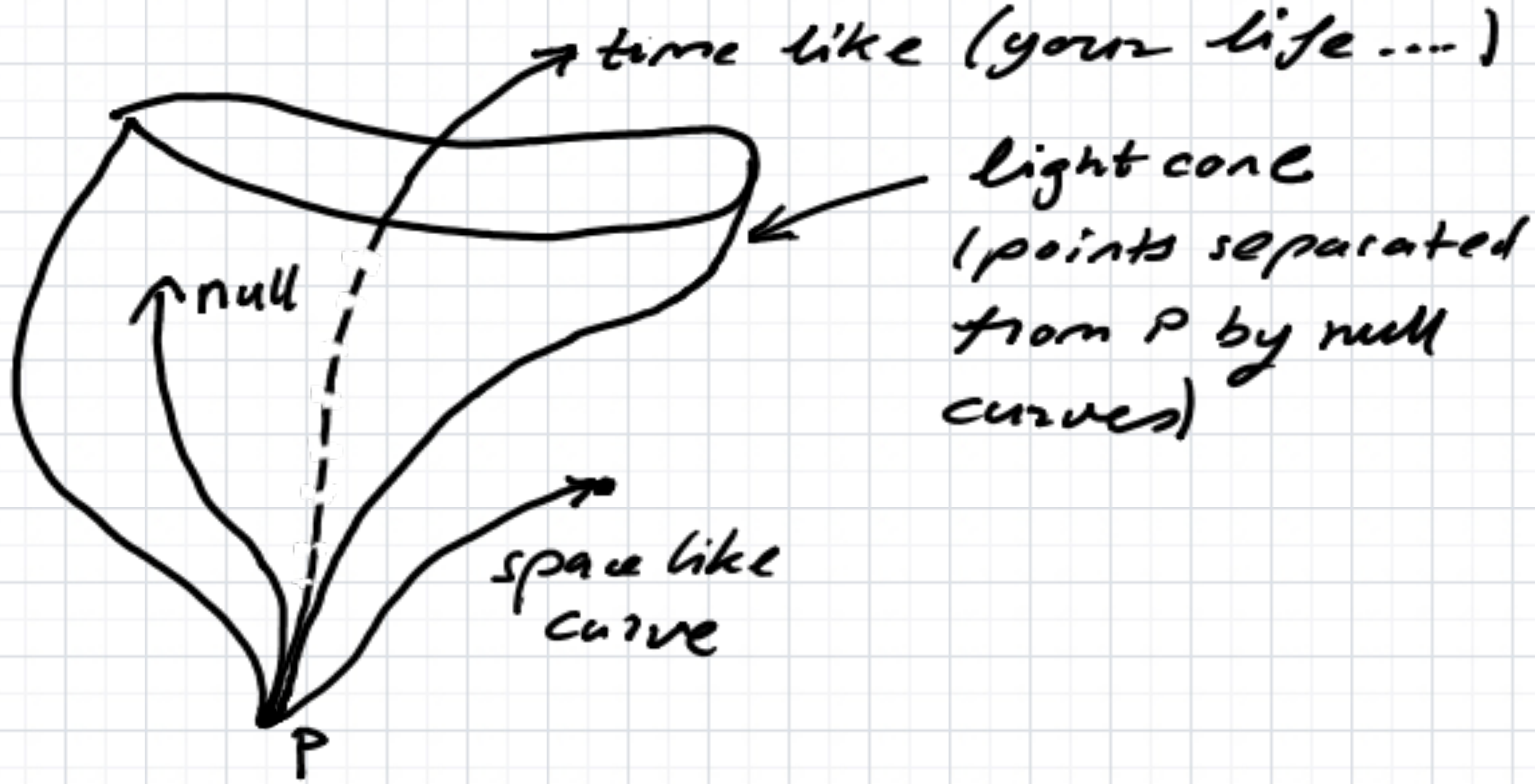
We call a vector v

(i) timelike $g(v,v) > 0$

(ii) spacelike $g(v,v) < 0$

(iii) null/light like $g(v,v) = 0$

Then we say curves are timelike, spacelike or null if their tangent vectors are everywhere timelike, space like or null.



An inner product induces a canonical isomorphism b/w a vector space and its dual:

$$\begin{tikzcd}
v \arrow[r, mapsto, "\iota"] \arrow[d, phantom, sloped, "\in"] & g_p(v,\cdot) \arrow[d, phantom, sloped, "\in"] \\
T_pM & T_p^*M \\
\binom{1}{0}\text{-tensor} \arrow[u, squiggly] & \binom{0}{1}\text{-tensor} \arrow[u, squiggly]
\end{tikzcd}$$

(contravariant vector) (covariant vector)

Notice this is the pullback again

$$\begin{tikzcd}
T_pM \arrow[r, "\iota"] \arrow[dr, "v^* = v\circ\iota"'] & T_p^*M \arrow[d, "v"] \\
& \mathbb{R}
\end{tikzcd}$$

If $\{e_i\}$ is a basis for T_pM and $\{\theta^i\}$ the dual basis for T^*_pM. Then

$$\iota(e_i) = g(e_i, \cdot) = w_{ij}\,\theta^j \quad \text{for some } w_{ij}$$

To compute w_{ij} notice

$$(\underbrace{\iota(e_i)}_{T^*_pM})(e_j) = g(e_i, e_j) = g_{ij}$$

but

$$(w_{ik}\,\theta^k)(e_j) = w_{ik}\,\theta^k(e_j) = w_{ik}\,\delta^k_j = w_{ij}$$

$$\Rightarrow \quad \iota(e_i) = g_{ij}\,\theta^j$$

so

$$\iota : e_i \longmapsto g_{ij}\,\theta^j \qquad \text{"\underline{lowering an index}"}$$

Similarly

$$\iota^{-1} : \theta^i \longmapsto g^{ij} e_j \qquad g^{ij} = (g_{ij})^{-1} \qquad \text{"\underline{raising an index}"}$$

In general, we can now biject tensors of types $\binom{n}{m}$ to types $\binom{0}{n+m}, \binom{0}{n+m-1}, \ldots, \binom{n+m}{0}$ using ι and ι^{-1}.

For example

$$T = T^{ij}{}_k \, e_i \otimes e_j \otimes \theta^k \xmapsto{\iota \otimes \iota \otimes \iota^{-1}} T_{ij}{}^k \, \theta^i \otimes \theta^j \otimes e_k$$

<u>Representations</u> The symmetric group can be used to break $gl(\dim M)$ representations into irreducibles. To further decompose into $so(\dim M)$ representations, the metric can be used:

<u>EX</u>
$$X_{ij} = X_{[ij]} + X_{(ij)} \quad \text{where} \quad \begin{cases} [1 \cdots n] = \frac{1}{n!} \sum_\sigma \operatorname{sgn}(\sigma)\, \sigma(1) \cdots \sigma(n) \\ (1 \cdots n) = \frac{1}{n!} \sum_\sigma \sigma(1) \cdots \sigma(n) \end{cases}$$

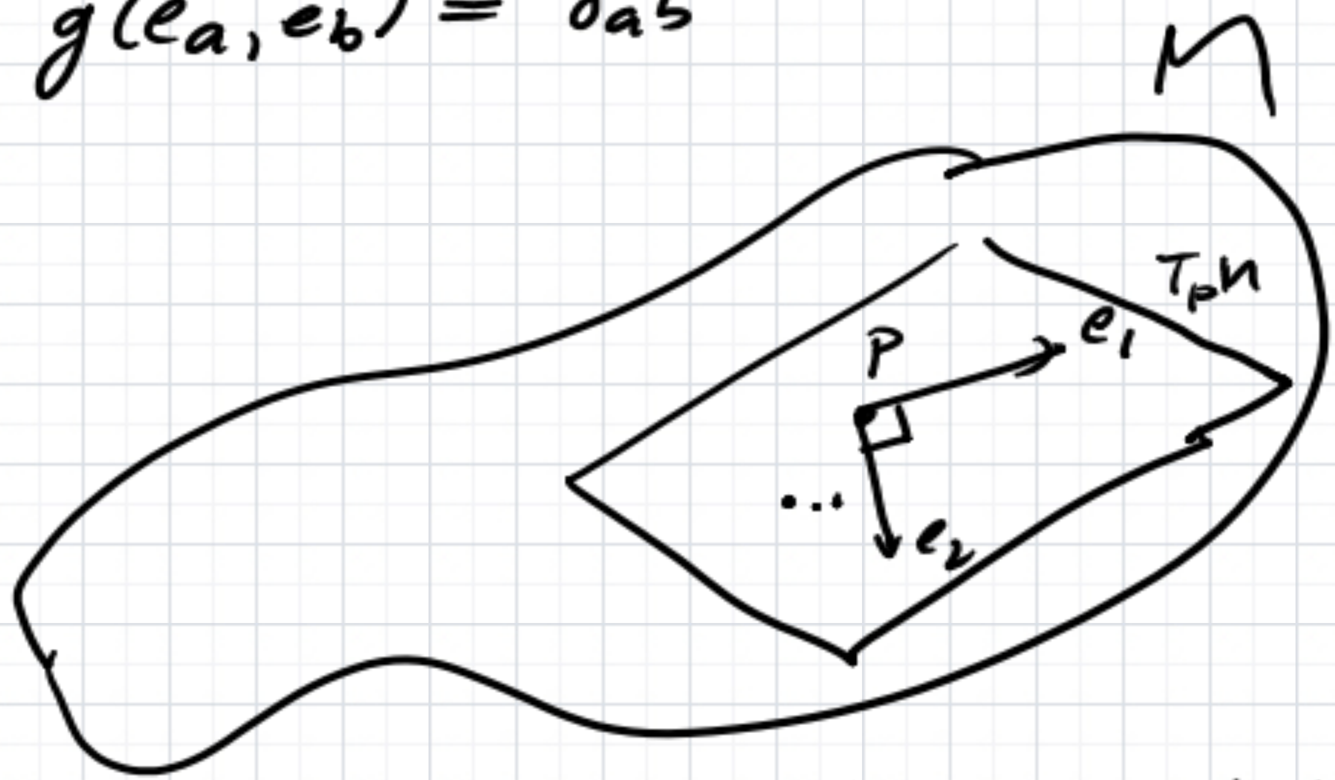
$$= X_{[ij]} + X_{\widetilde{ij}} + \frac{1}{\dim M} g_{ij} \, g^{kl} X_{kl}$$

Coordinate vs. non-coordinate bases

An orthonormal frame is a set of n-vector fields $\{e_a \in \mathfrak{X}M\}$ such that

$$g(e_a, e_b) = \delta_{ab}$$

i.e.



e_a are orthonormal basis for T_pM at each $p \in M$.

Can always construct an orthonormal frame for an open set $U \subset M$, but globally there is an obstruction measured by 2nd Stiefel-Whitney class.

$\underline{\underline{Ex}}$ $\mathbb{R}^2$ with $ds^2 = dx^2 + dy^2$.

$$\left\{ e_x = \frac{\partial}{\partial x} \;,\; e_y = \frac{\partial}{\partial y} \right\}$$ is an orthonormal frame because

$$g(e_x, e_x) = (dx \otimes dx)(e_x, e_x) = 1$$

$$g(e_x, e_y) = 0$$

$$g(e_y, e_y) = (dy \otimes dy)(e_y, e_y) = 1$$

$\underline{\text{Now}}$ in general if $\{ \frac{\partial}{\partial x^i} \in \mathfrak{X}U \}$ $U \subset M$ (U open) and $\{e_a\}$ is an orthonormal frame on U, then

$$e_a = e_a{}^i \frac{\partial}{\partial x^i}$$

components of vectors e_a in basis ∂_i

But $\delta_{ab} = g(e_a, e_b) = e_a{}^i e_b{}^j g(\partial_i, \partial_j)$

$$= e_a{}^i e_b{}^j g_{ij}$$

$$\Longrightarrow \quad \boxed{e_a{}^i e_b{}^j g_{ij} = \delta_{ab}}$$ The functions $e_a{}^i$ are often called the inverse "vielbein" (Gell-Mann).

<u>Notice</u>: Even though the metric components in the basis $\{e_a\}$ are just δ_{ab} (b/c $g_{ab} = g(e_a, e_b) = \delta_{ab}$), this basis is usually <u>not</u> a coordinate basis.

i.e. $$\mathcal{L}_{e_a} e_b = [e_a, e_b] = \underbrace{c_{ab}{}^{c} e_c}_{\text{b/c } \{e_a\} \text{ is a basis}}$$

and the structure functions $c_{ab}{}^{c}$ are in general not zero while $\mathcal{L}_{\partial_i}(\partial_j) = [\partial_i, \partial_j] = 0$ in a coordinate basis.

Review Exercise ① Show that the vielbein

$e_i{}^a = (e_a{}^i)^{-1}$ is the "square root" of

the metric tensor in the sense

$$\boxed{e_i{}^a e_j{}^b \delta_{ab} = g_{ij}}$$

② Consider the metric $ds^2 = dr^2 + r^2 d\theta^2$

for $\mathbb{R}^2$. Construct vielbeine for this metric.

Is this a coordinate basis?