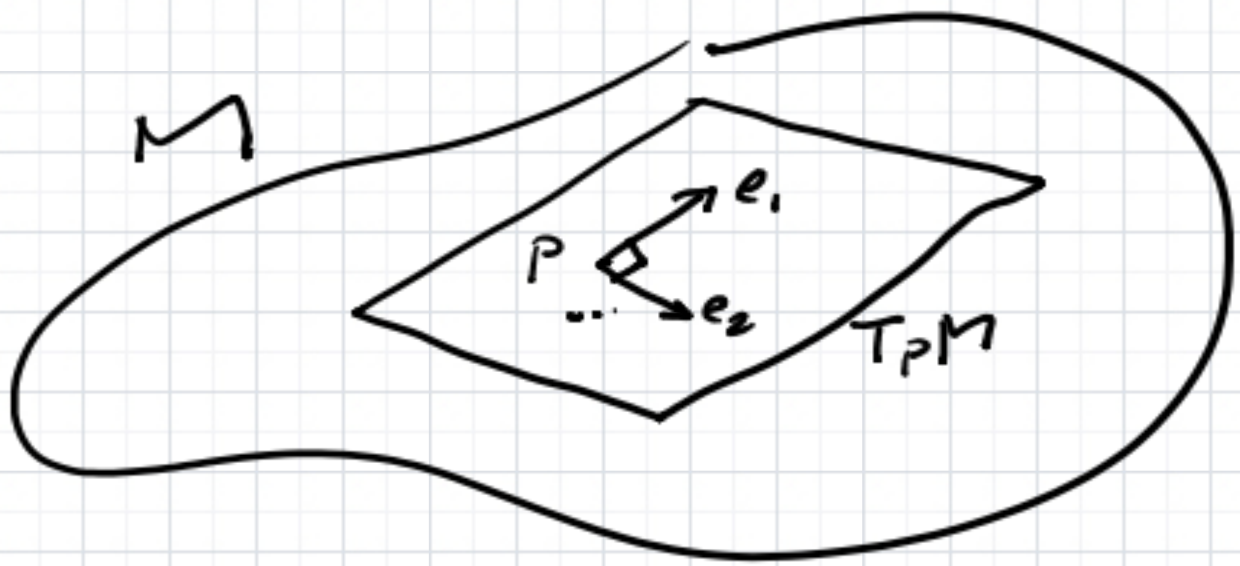
Lecture # 15 Orthonormal Frame

Let $U \subset M$ open set and study orthonormal frames

$$\{ e_a \mid e_a \in \mathfrak{X}M,\ g(e_a, e_b) = \delta_{ab} \}$$

i.e. choose an orthonormal basis for each T_pM



Here we are assuming g is proper. Else require

$$g(e_a, e_b) = \eta_{ab} = \mathrm{diag}(\underbrace{-, \dots, -}_{p}, \underbrace{+, \dots, +}_{q})$$

which is the invariant tensor of $SO(p,q)$ $\quad p+q = \dim M$

EX $\mathbb{R}^2$ with $ds^2 = dx^2 + dy^2$ $\quad \{ e_x = \frac{\partial}{\partial x}, e_y = \frac{\partial}{\partial y} \}$

is an orthonormal frame.

Now if $\left\{\frac{\partial}{\partial x^i}\right\}$ are coordinate vector fields and $\{e_a\}$ an orthonormal frame we can write

$$e_a = e_a{}^i \frac{\partial}{\partial x^i}$$

Here the components of the vector field e_a in the basis $\frac{\partial}{\partial x^i}$ are a $GL(\dim M)$-valued matrix.

Their inverses $(e_a{}^i)^{-1} = e_i{}^a$ were "dubbed "<u>vielbeine</u>" by Gell-Mann satirizing their earlier four dimensional title "vierbein" or "tetrad".

Notice

$$\delta_{ab} = g(e_a, e_b) = g\left(e_a{}^i \frac{\partial}{\partial x^i}, e_b{}^j \frac{\partial}{\partial x^j}\right)$$

$$= e_a{}^i e_b{}^j g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = e_a{}^i e_b{}^j g_{ij}$$

$$\Rightarrow \quad \boxed{g_{ij} = e_i{}^a e_j{}^b \delta_{ab}}$$

so the vielbeine are a "square root of the metric".

This is example of an old trick, namely the space of symmetric invertible matrices is the coset $Gl(n)/SO(n)$ because

$$\underbrace{M}_{\text{symmetric}} = \underbrace{E^T E}_{Gl(n)\text{-valued}} = (OE)^T OE \quad \text{so} \quad E \sim OE.$$

with $O \in O(n)$.

Notice that an orthonormal frame is in general not a coordinate basis. Rather

$$[e_a, e_b] = c_{ab}{}^c e_c$$

for some functions $c_{ab}{}^c$. [Looks a lot like the structure constants of a Lie algebra].

Example $\mathbb{R}^2$ with polar coordinates $ds^2 = dr^2 + r^2 d\theta^2$.

$$\left\{ e_1 = \frac{\partial}{\partial r}, \; e_2 = \frac{1}{r}\frac{\partial}{\partial \theta} \right\} \text{ is an orthonormal basis.}$$

$g(e_1,e_1) = dr \otimes dr\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1$, $g(e_1,e_2) = 0$, $g(e_2,e_2) = \frac{r^2}{r^2} d\theta \otimes d\theta\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = 1$

but $[e_1, e_2] = -\frac{1}{r^2}\frac{\partial}{\partial \theta} = -\frac{1}{r} e_2$ (even though ds^2 is flat).

Isometries

Question: when are Riemannian structures equivalent?

Let (M,g) & (N,h) be Riemannian manifolds.

We call a diffeomorphism

$$f : M \longrightarrow N$$

an isometry (and (M,g), (N,h) isometric) if

$$(df \otimes df)^* \circ h = g$$

$$\begin{tikzcd} \otimes^2 TM \arrow[r, "df \otimes df"] \arrow[dr, bend right, "g"'] \arrow[r, phantom, shift right=4ex, "\circlearrowleft"] & \otimes^2 TN \arrow[d, "h"] \\ & \mathbb{R} \end{tikzcd}$$

So, for vectors $v, w \in T_pM$

$$\boxed{g(v,w)_p = h(df(v), df(w))_{f(p)} \quad \forall\, p \in M}$$

Local isometries require only that a neighborhood $U \ni p$ be isometric to $f(U)$.

Note We can form a group from isometries by taking $(\mathcal{U}, g) \xrightarrow[f]{} (\mathcal{U}, h)$ and then using composition of maps for group multiplication.

Killing Vectors

The integral curves $\gamma(t, \cdot)$ of a differentiable vector field X form a local group of diffeomorphisms

$$\gamma(t, \cdot) \circ \gamma(t', \cdot) = \gamma(t+t', \cdot).$$

We say that a vector field X is Killing if its integral curves $\gamma(t, \cdot)$ are local isometries.

Our previous condition for an isometrie really said

$$f^*(h) = g$$

and we already found how to infinitesimally compare pull backs of diffeomorphisms — the Lie derivatives.

Hence, for X to be Killing we must require

$$\boxed{\mathcal{L}_X g = 0}$$ Killing's equation

In components this says

$$\mathcal{L}_{X^i\partial_i} g_{kl}\, dx^k dx^l = [X^i\partial_i, g_{kl}\, dx^k dx^l]$$

$$= X^i(\partial_i g_{kl})\, dx^k dx^l + 2 g_{kl}\, dX^k dx^l$$

$$= (X^i \partial_i g_{kl} + 2 g_{il}\, \partial_k X^i)\, dx^k dx^l$$

$$= (X^i \partial_i g_{kl} + 2 g_{il}\, \partial_k (g^{ij} X_j))\, dx^k dx^l$$

$$= (2\,\partial_k X_j - 2\,\partial_k g_{il} X^i + \partial_i g_{kl} X^i)\, dx^k dx^l$$

Using the Christoffel symbols we found before

$$= 2[\partial_k X_j - 2\Gamma^i_{kj} X_i]\, dx^k dx^l$$

$$= [\nabla_k X_j + \nabla_j X_k]\, dx^k dx^l$$

where $\boxed{\nabla_k X_j = \partial_k X_j - \Gamma^i_{kj} X_i}$

is the <u>covariant derivative</u> on a 1-form and

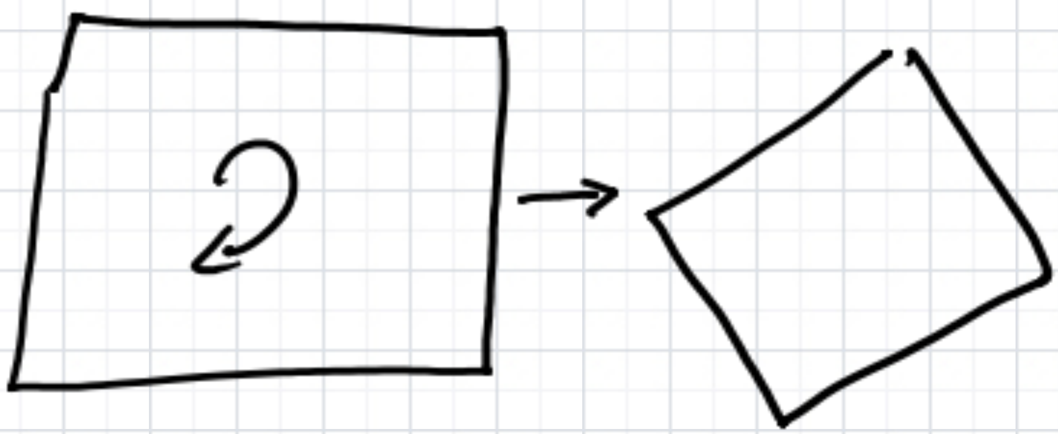
$\boxed{0 = \nabla_k X_j + \nabla_j X_k}$ is Killing's equation.

Ex $\frac{\partial}{\partial\theta} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ is Killing on $(\mathbb{R}^2, dx^2 + dy^2)$

because $[\frac{\partial}{\partial\theta}, dx^2 + dy^2] = 2dx\, d[(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})x]$

$$+ 2dy\, d[(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})y]$$

$$= -2dx\,dy + 2dy\,dx = 0.$$



rotating the plane w/ its standard metric about the origin gives same Riemannian structure.

Exercise: Suppose $f: M \to N$ is a differentiable mapping. Discuss how to obtain a Riemannian structure on M from one on N.

As an example let $N = \mathbb{R}^3$ and $M = S^2$, the unit sphere at the origin.