Lecture #16 Isometries

We said $X \in \mathfrak{X}M$ was Killing if its integral curves $\gamma(t,\cdot)$ generated local isometries. In equations

$$\boxed{\mathcal{L}_X g = 0}$$

Now

$$\mathcal{L}_{X^k \frac{\partial}{\partial x^k}} g_{ij}\, dx^i dx^j = X^k \frac{\partial g_{ij}}{\partial x^k} dx^i dx^j + 2 g_{ij}\, dX^j dx^i$$

$$= dx^i dx^j \left(X^k \frac{\partial}{\partial x^k} g_{ij} + 2 g_{ik} \frac{\partial X^k}{\partial x^j} \right)$$

$$= dx^i dx^j \left(X^k \frac{\partial}{\partial x^k} g_{ij} + 2 g_{ik} \frac{\partial}{\partial x^j} (g^{kl} X_l) \right)$$

Using the christoffels found earlier

$$= dx^i dx^j \left(X_k g^{kl} \frac{\partial}{\partial x^l} g_{ij} - 2 X_k g^{kl} \frac{\partial}{\partial x^j} g_{il} + 2 \frac{\partial}{\partial x^j} X_i \right)$$

$$= 2\, dx^i dx^j \left(\partial_j X_i - \Gamma^k_{ji} X_k \right)$$

$$= dx^i dx^j \left(\nabla_i X_j + \nabla_j X_i \right) \quad \text{where } \boxed{\nabla_i X_j \equiv \frac{\partial}{\partial x^i} X_j - \Gamma^k_{ij} X_k}$$

$$\Longrightarrow \quad \boxed{\nabla_i X_j + \nabla_j X_i = 0} \quad \text{Killing's Equation}$$

<u>EX</u> $\left(\mathbb{R}^n, (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2\right)$ has Killing vectors

$$\frac{\partial}{\partial x^1}, \quad \cdots \quad, \frac{\partial}{\partial x^n} \qquad\qquad n$$

$$x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}, \quad \cdots, \quad x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \quad \cdots \qquad\qquad \frac{n(n-1)}{2}$$

$$\overline{\qquad\qquad\qquad}$$

"Maximally symmetric space" $\longrightarrow$ $\dfrac{n(n+1)}{2}$

$\dfrac{\partial}{\partial x^i}$ = translations in the direction i

$x^i \dfrac{\partial}{\partial x^j} - x^j \dfrac{\partial}{\partial x^i}$ = rotations in ij-plane

<u>Notice</u> $\left[x^i \dfrac{\partial}{\partial x^j} - x^j \dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^k}\right] = \delta^j_k \dfrac{\partial}{\partial x^i} - \delta^i_k \dfrac{\partial}{\partial x^j}$

$$\left[x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, x^k \frac{\partial}{\partial x^l} - x^l \frac{\partial}{\partial x^k}\right] = \delta^k_j \left(x^i \frac{\partial}{\partial x^l} - x^l \frac{\partial}{\partial x^i}\right)$$

$\pm$ 3 more

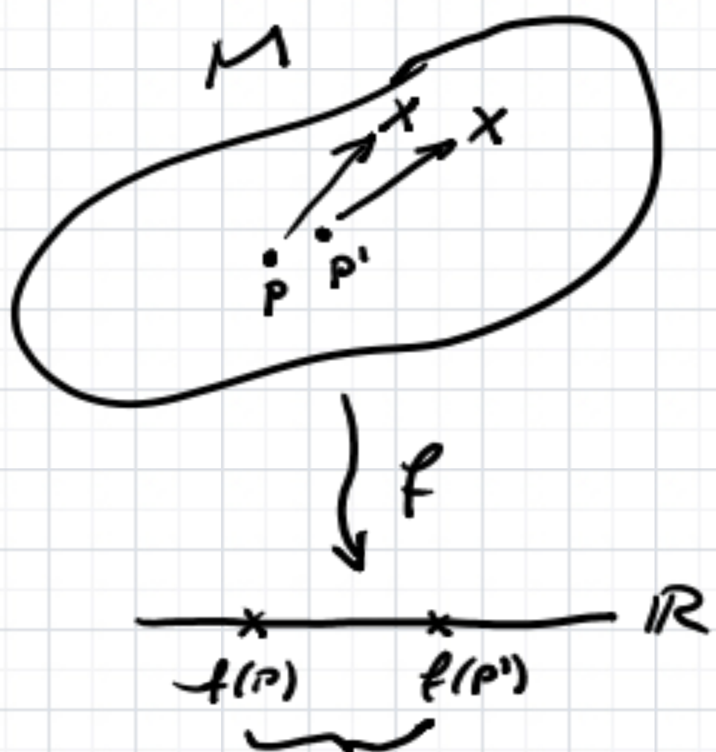
This is the Lie algebra of the Euclidean group.

Directional Derivatives

Given a function f, the 1-form df corresponds to its <u>gradient</u>. To compute the directional derivative of f in the direction of a vector field X we just act with df on X

$$\nabla_X f \equiv \langle df, X \rangle \underset{\text{defn of vectors}}{=} X(f) \underset{\text{Lie derivative}}{=} \mathcal{L}_X(f) = [X, f]$$

Notice



Δf makes sense b/c $f(p)$ & $f(p')$ are in the same space.

What about directional derivatives of vector or tensors?

Consider the flow in $\mathbb{R}^2$ along $\frac{\partial}{\partial x}$

In vector calculus we would write $\frac{\partial}{\partial x} = (1,0,0)\cdot\vec{\nabla}$

and in any direction $\vec{X}$ compute $\vec{X}\cdot\vec{\nabla}(1,0,0) = \vec{0}$

which makes sense. This flow is always in the same direction everywhere in the plane.

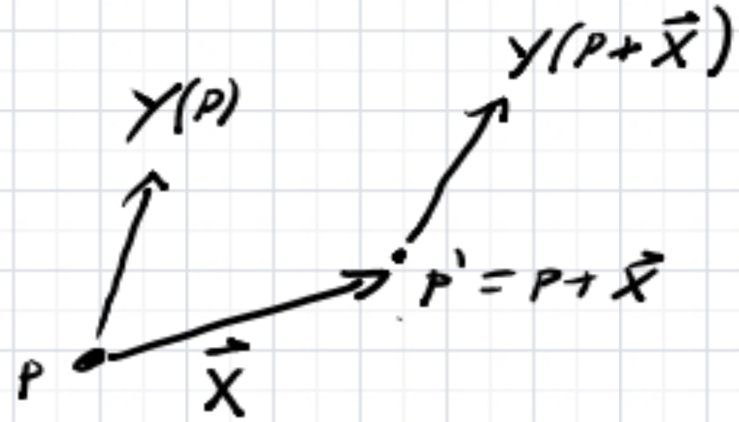
<u>Notice</u> therefore that the directional derivative could not be the Lie derivative. For example take

$$\vec{X} = (-y, x) \Rightarrow \vec{X}\cdot\vec{\nabla} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial \theta}$$

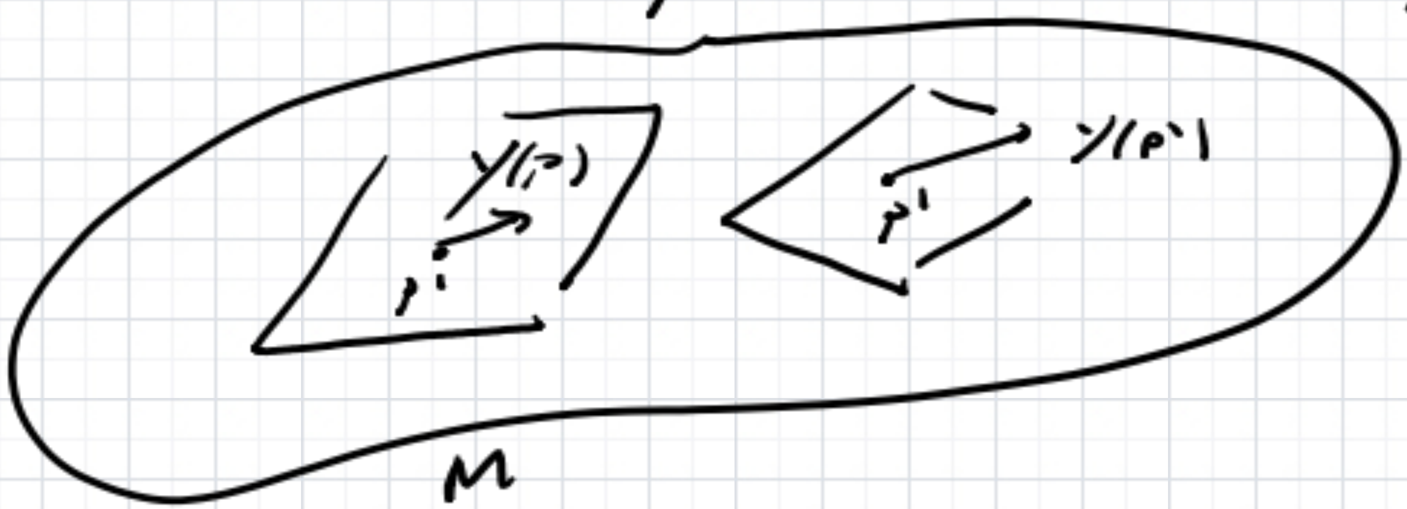
Then $$\mathcal{L}_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial x} = \left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right] = \frac{\partial}{\partial y} \neq 0$$

This is because $\mathcal{L}_{\frac{\partial}{\partial\theta}}$ drags $\frac{\partial}{\partial x}$ around the curve $r = \text{constant}$.

The Euclidean directional derivative, instead compared



but this is not possible on a manifold because



$Y(p)$ and $Y(p')$ are in different spaces.

What the Euclidean examples do is assume a notion of parallelism.

Really need a gradient operator

$$\underset{\text{vector fields}}{\nabla : \mathfrak{X}M} \longrightarrow \underset{\binom{1}{1}\text{-tensor fields}}{\Gamma(T^1_1 M)}$$

Then $\nabla : u \longmapsto \nabla u$ and the directional derivative is $\nabla_X u = (\nabla u)(X, \cdot)$

$\cdot$ ← slot for 1-forms.

so that $\nabla_{X^i \frac{\partial}{\partial x^i}} u^j \frac{\partial}{\partial x^j} = (X^i \underbrace{\nabla_i u^j}) \frac{\partial}{\partial x^j}$

components of the (1,1) tensor ∇u.

contrast with

$$\mathcal{L}_X u = (X^i \partial_i u^j - u^i \partial_i X^j) \frac{\partial}{\partial x^j}$$

<u>Analyze $\nabla_i u^j$.</u>

Ansatz $\nabla_i u^j \stackrel{?}{=} \partial_i u^j$

<u>Jacobian</u>

But under a change of coordinates $u^j = \frac{\partial x^j}{\partial x'^i} u'^i$

$$\Rightarrow \frac{\partial u^j}{\partial x^i} = \frac{\partial x'^k}{\partial x^i} \frac{\partial}{\partial x'^k} \left(\frac{\partial x^j}{\partial x'^l} u'^l \right)$$

Chain rule

$$= \underbrace{\frac{\partial x'^k}{\partial x^i} \frac{\partial x^j}{\partial x'^l} \frac{\partial u'^l}{\partial x'^k}}_{\text{Correct transformation of (1,1)-tensor}} + \underbrace{\frac{\partial x'^k}{\partial x^i} \frac{\partial^2 x^j}{\partial x'^k \partial x'^l} u'^l}_{\text{Bad stuff!}}$$

Improved ansatz

Not tensor

$$\underbrace{\nabla_i u^j}_{\text{tensor}} = \frac{\partial u^j}{\partial x^i} + \Gamma^j_{ik} u^k$$

"Connection coefficients"

Require
$$\boxed{\Gamma^j_{ik} = \frac{\partial x^j}{\partial x'^{j'}} \frac{\partial x'^{i'}}{\partial x^i} \frac{\partial x'^{k'}}{\partial x^k} \Gamma'^{j'}_{i'k'} - \frac{\partial x'^{l}}{\partial x^i} \frac{\partial^2 x^j}{\partial x'^{l} \partial x'^{k}}}$$

Claim <u>a</u> solution is

$$\boxed{\Gamma^j_{ik} = \frac{1}{2} g^{jl} \left(\partial_i g_{lk} + \partial_k g_{li} - \partial_l g_{ik} \right)}$$

"Riemannian / Affine Connection coefficients".

<u>Review Exercise</u>: Check the above Claim.