Lecture # 17 The Gradient

Under a change of coordinates a vector $u = u^i \partial_i$ obeys

$$u = u^i(x) \frac{\partial}{\partial x^i} = u'^j(x') \frac{\partial}{\partial x'^j} = u'^j(x') \underbrace{\frac{\partial x^i}{\partial x'^j}}_{\text{Jacobian}} \frac{\partial}{\partial x^i}$$

$$\Rightarrow u^i = \frac{\partial x^i}{\partial x'^j} u'^j$$

$$\Rightarrow \frac{\partial u^i}{\partial x^k} = \frac{\partial x'^l}{\partial x^k} \frac{\partial}{\partial x'^l} \left(\frac{\partial x^i}{\partial x'^j} u'^j \right)$$

$$= \underbrace{\frac{\partial x'^l}{\partial x^k} \frac{\partial x^i}{\partial x'^j} \frac{\partial u'^j}{\partial x'^l}}_{\text{TRANSFORMATION OF A } \binom{1}{1}\text{-TENSOR}} + \underbrace{\frac{\partial x'^l}{\partial x^k} \frac{\partial^2 x^i}{\partial x'^l \partial x'^j} u'^j}_{\text{NOT TENSOR}}$$

So modify

$$\nabla u = \left(\partial_i u^j + \Gamma^j_{ik} u^k \right)$$

Where

$$T^{i}_{jk} = \underbrace{\frac{\partial x^{i}}{\partial x'^{l}} \frac{\partial x'^{m}}{\partial x^{j}} \frac{\partial x'^{n}}{\partial x^{k}} T'^{i}_{jk}}_{\binom{1}{2}\text{-TENSOR TRANSFORMATION}} - \frac{\partial x'^{m}}{\partial x^{j}} \frac{\partial^2 x^{i}}{\partial x'^{m} \partial x'^{n}} \frac{\partial x'^{n}}{\partial x^{k}}$$

Note if $J = \left(\frac{\partial x}{\partial x'}\right)$ is the Jacobian and we call $T = (T^{i}_{k}) = dx^{j}\, T^{i}_{jk}$ then

$$\boxed{T = J^{-1} T J - J^{-1} dJ}$$

or even sweeter

$$T - d = J^{-1}(T - d)J$$

as an operator statement.

This is called the transformation of a $\mathfrak{gl}(\dim M)$-valued connection.

A (particular) solution for T is the Christoffel symbols

$$\boxed{T^i_{jk} = \frac{1}{2} g^{il} \left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk} \right)}$$

which are not the components of a $\binom{1}{2}$-tensor thanks to the partial derivatives of g.

We can add any $\binom{1}{2}$-tensor to Γ and obtain a solution for T transforming correctly (homogeneous solution).

When $T = \Gamma$, we call ∇ the affine connection, else we say that ∇ is torsionful.

Torsion can result in two evils

* Failure of parallelograms to close
* Failure of metric to be parallel.

Example: You have actually been using a torsion-free covariant derivative ∇_μ where $\nabla = dx^\mu \nabla_\mu$ for a long time without knowing it.

This is because in Cartesian coordinates $\Gamma^{i}_{jk} = 0$.

Even in polar coordinates this is no longer so:

$(\mathbb{R}^2, ds^2 = dr^2 + r^2 d\theta^2)$

$$dr^2 = dr\, g_{rr}\, dr + d\theta\, g_{\theta\theta}\, d\theta$$

$$\Rightarrow \begin{cases} g_{rr} = 1 \\ g_{\theta\theta} = r^2 \end{cases} \Rightarrow (g_{ij}) = \begin{pmatrix} 1 & \\ & r^2 \end{pmatrix} \Rightarrow (g^{ij}) = \begin{pmatrix} 1 & \\ & \frac{1}{r^2} \end{pmatrix}$$

Non-vanishing Christoffels:

$$\Rightarrow 2\Gamma_{r\theta\theta} = 2\partial_\theta g_{r\theta} - \partial_r g_{\theta\theta} = -2r \Rightarrow \Gamma^r_{\theta\theta} = -r$$

$$2\Gamma_{\theta r\theta} = \partial_r g_{\theta\theta} + \partial_\theta g_{r\theta} - \partial_\theta g_{\theta r} = \partial_r r^2 = 2r$$

$$\Rightarrow \Gamma^\theta_{r\theta} = \frac{1}{r}$$

Lets compute the <u>Laplacian</u>. Consider a function f.

Then $\nabla f \equiv df$ is a 1-form $\Rightarrow g^{-1}(\nabla f, \cdot)$ is a vector.

Explicitly $g^{-1}(\nabla f, \cdot) = \left(\frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta} \otimes \frac{\partial}{\partial \theta}\right)\left(\frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta\right)$

$$= \frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta} \quad \underline{\text{A VECTOR}}$$

Now ∇ : vectors $\longmapsto$ $\binom{1}{1}$-tensors

But there is a canonical method to produce a scalar from a $\binom{1}{1}$-tensor called trace

$$T_j^i \, dx^j \otimes \frac{\partial}{\partial x^i} \xrightarrow{\text{trace}} T_j^i \left\langle dx^j, \frac{\partial}{\partial x^i} \right\rangle = T_i^i$$

For a general vector u, $\operatorname{trace}(\nabla u) = \operatorname{div}(u) = \nabla_i u^i$

We call $\operatorname{div}(g^{-1}(\nabla f, \cdot)) = \Delta f = \nabla_i \nabla^i f$ the <u>Laplacian</u> of f.

Here $\operatorname{div}(u) = \nabla_r u^r + \nabla_\theta u^\theta$

$$= \partial_r u^r + \cancel{\Gamma^r_{rk} u^k}^{\,0} + \partial_\theta u^\theta + \Gamma^\theta_{\theta r} u^r$$

$$= \partial_r u^r + \partial_\theta u^\theta + \frac{1}{r} u^r$$

But putting $(u^r, u^\theta) = (\partial_r f, \frac{1}{r^2}\partial_\theta f)$ gives

$$\Delta f = \partial_r^2 f + \frac{1}{r^2}\partial_\theta^2 f + \frac{1}{r}\partial_r f$$
$$= \frac{1}{r}\partial_r(r\,\partial_r f) + \frac{1}{r^2}\partial_\theta^2 f$$

The standard Laplacian in Polar coordinates

Note that $\Delta f = \operatorname{div}(g^{-1}(\nabla f, \cdot))$ generalizes to any tensor and is called the Bochner Laplacian.

LINEAR CONNEXION / GRADIENT

$$\begin{array}{ccc} \nabla : \Gamma(T^p_q M) & \longrightarrow & \Gamma(T^p_{q+1} M) \\ \Psi & & \Psi \\ T & \longmapsto & \nabla T \end{array}$$

such that
(i) $\nabla(T+S) = \nabla T + \nabla S$
(ii) $\nabla(fT) = df \otimes T + f\nabla T \qquad f: M \to \mathbb{R}$
(iii) $\nabla(T \otimes S) = \nabla T \otimes S + T \otimes \nabla S$
(iv) $\nabla\langle \omega, w\rangle = \langle \nabla_v \omega, w\rangle + \langle \omega, \nabla_v w\rangle$

If $\{e_a\}$, $\{\theta^b\}$ are dual, but not necessarily orthonormal bases

$$\boxed{\nabla e_a \equiv \gamma^c_{\ a b}\, \theta^b \otimes e_c}$$

where the <u>functions</u> $\gamma^l_{\ ki}$ are called <u>connection coefficients.</u>

* When $e_a = \frac{\partial}{\partial x^a}$, coordinate vector fields, and ∇ is torsion free, $\gamma^c_{\ ab} = \Gamma^c_{ab}$, the <u>Christoffel symbols.</u>

* $\gamma^l_{\ ki}$ are <u>NOT</u> tensor components.