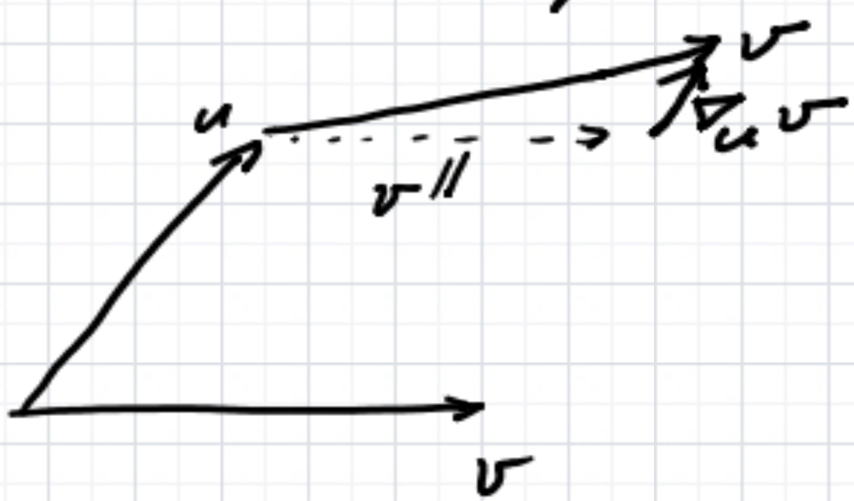
# Lecture #19 — TORSION & CURVATURE

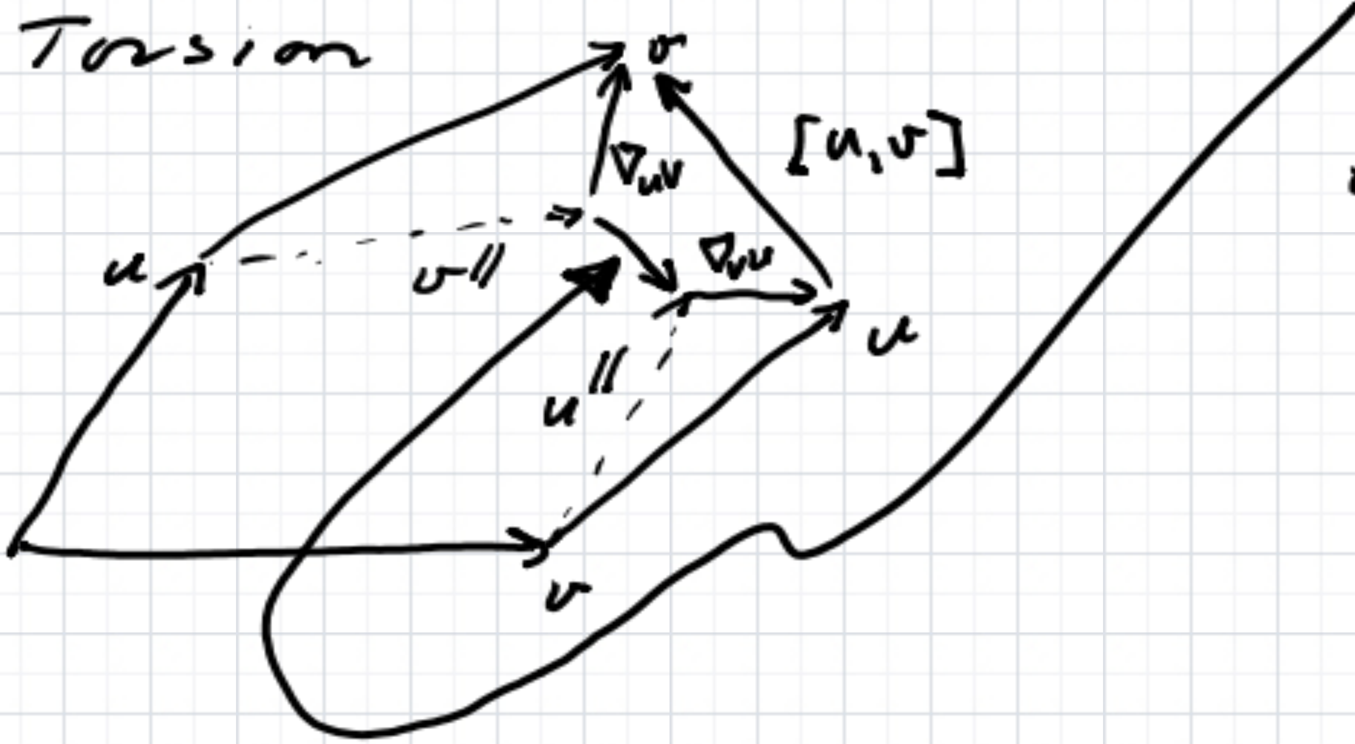
The bracket closes quadrilaterals


Can make this picture precise using integral curves

Parallel transport




Torsion




vector

$$T(\underbrace{\cdot}_{\text{eats 1 form}},u,v) = \nabla_u v - [u,v] - \nabla_v u$$

## TORSION TENSOR $\binom{1}{2}$-tensor

$$T(\alpha, u, v) = \langle \alpha, \nabla_u v - \nabla_v u - [u, v] \rangle$$

1-form vectors

* $T(\alpha, u, v) = T(\alpha, v, u) \Rightarrow T(\alpha, \cdot, \cdot) \in \Lambda^2 M$

**Components**

coordinate quadrilaterals close

$$T\left(dx^j, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \langle dx^j, \nabla_{\partial_i} \partial_k - \nabla_{\partial_k} \partial_i - \cancel{[\partial_i, \partial_k]} \rangle$$

$$= \langle dx^j, \Gamma^l_{ik} \partial_l - \Gamma^l_{ki} \partial_l \rangle$$

$$= \Gamma^j_{ik} - \Gamma^j_{ki}$$

$$\Rightarrow \boxed{T^j_{ij} = \Gamma^j_{ik} - \Gamma^j_{ki}}$$

Anti-symmetric part of christoffels.

* $\Gamma^j_{[ik]}$ are components of a tensor.

* Recall, in vector calculus, $\vec{v} \times \vec{w}$ defined a parallelogram, the same is true in general for 2-forms (they can be defined as objects that can be integrated over surfaces). So the torsion assigns a vector to infinitesimal pieces of surface.

* Notice

$$\Gamma(g)^j_{ik} = \tfrac{1}{2} g^{jl}[\partial_i g_{kl} + \partial_k g_{il} - \partial_l g_{ik}]$$
$$= \Gamma(g)^j_{ki}$$

gives a torsion free connection.

In fact, you can solve the conditions

(i) $T=0$ torsion free

(ii) $\nabla g = 0$ covariantly constant metric.

and find (uniquely) $\Gamma(g)$ the <u>Levi Civita connexion</u>.

* Notice that $T(\bullet, u, v)$ is a vector so acts on functions via

$$T(\bullet,u,v)(f) = (\nabla_u \nabla_v - \nabla_u \nabla_v - \nabla_{[u,v]})(f)$$

$v(f) = \nabla_v(f)$

so computes the "algebra" of <u>covariant derivatives</u>.

# Curvature

Algebraically, this is the same as the torsion but acting on vectors rather than functions:

$$T(\cdot, u, v)(f) = \left(\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}\right)(f)$$

maps functions to functions

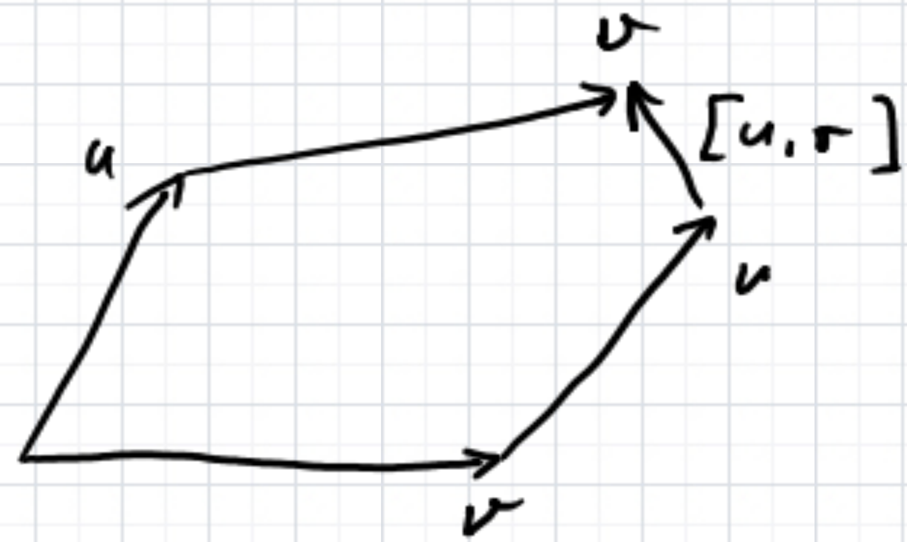
$$\underbrace{R(\overset{\uparrow}{w}, \overset{\uparrow}{\cdot}, u, v)}_{\text{vector}} = \left(\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}\right)(w)$$

input a vector

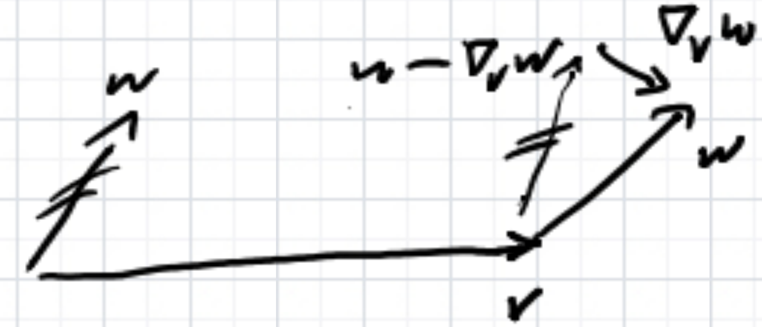
slot empty receives 1-forms

* Geometric interpretation is parallel transport around loops.

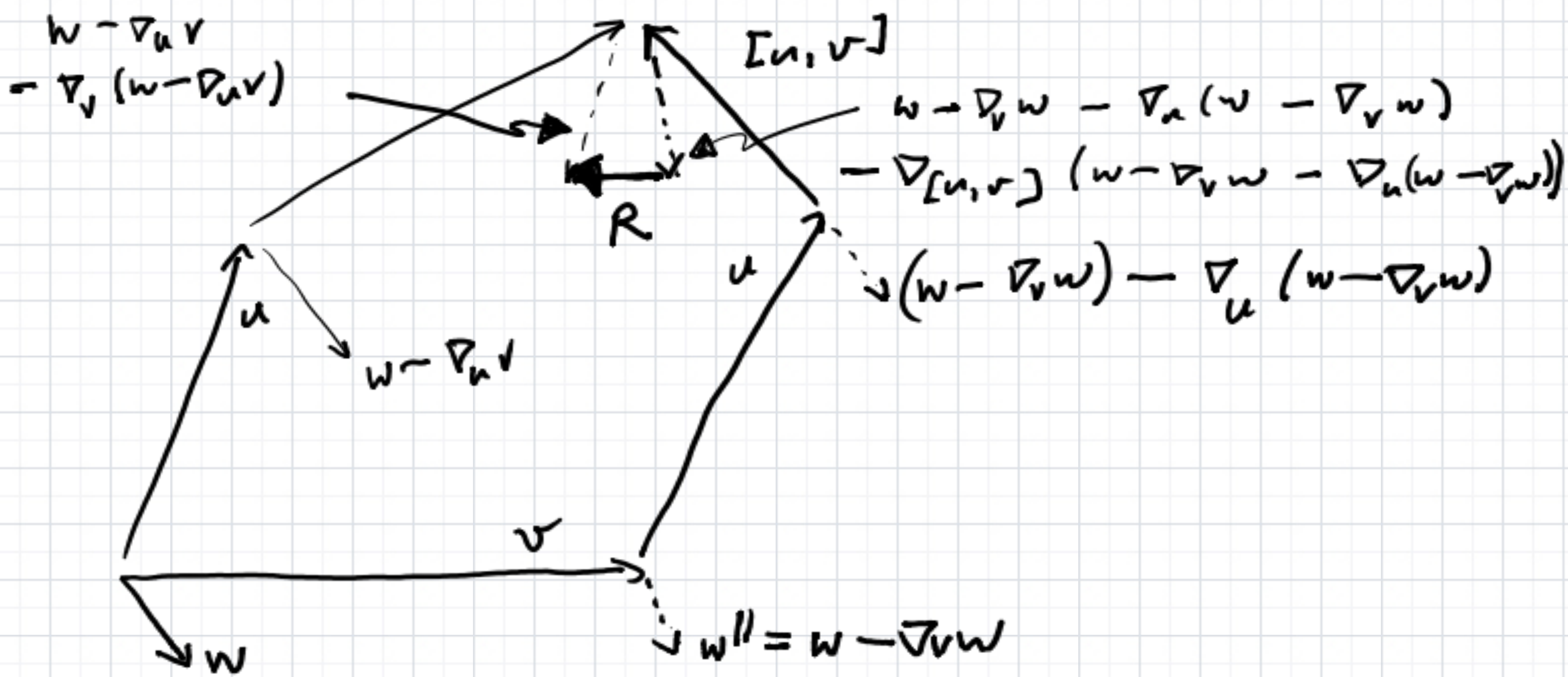
As before define a loop using $u, v, [u,v]$




and $\parallel$ transport




Now study parallel transport of $w$ around this loop:



The vector

$$R = \cancel{w} - \nabla_u w - \nabla_v (w - \nabla_u w)$$
$$- (\cancel{w} - \nabla_v w - \nabla_u w + \nabla_u \nabla_v w$$
$$- \nabla_{[u,v]}[w - \nabla_v w - \nabla_u w + \nabla_u \nabla_v w])$$

$$\doteq \nabla_v \nabla_u w - \nabla_u \nabla_v w - \nabla_{[u,v]} w \quad + \underline{\text{Higher order}}$$

<u>As above</u>

* $R$ is a $\binom{3}{1}$-tensor

* $R(w, \alpha, \cdot, \cdot) \in \Lambda^2 M$

* View $R$ as measuring rotation of $w$ per unit surface area.

# Computing The Riemann Tensor

$$R(\partial_i, dx^k, \partial_m, \partial_n) \equiv R_i{}^k{}_{mn}$$

$$\equiv \langle dx^k, \nabla_m \nabla_n (\partial_i) - \nabla_n \nabla_m (\partial_i) - \cancel{\nabla_{[\partial_m, \partial_n]} \partial_i} \rangle$$

call $\nabla_{\partial_m} = \nabla_m$

$$= \langle dx^k, \nabla_m \Gamma^l_{ni} \partial_l - \nabla_n \Gamma^l_{mi} \partial_l \rangle$$

functions

$$= \langle dx^k, (\partial_m \Gamma^l_{ni}) \partial_l - \Gamma^l_{ni} \nabla_m (\partial_l) - (m \leftrightarrow n) \rangle$$

$$= \partial_m \Gamma^k_{ni} - \partial_n \Gamma^k_{mi} - \Gamma^l_{ni} \Gamma^k_{ml} + \Gamma^l_{m} \Gamma^k_{nl}$$

$$\Rightarrow \boxed{R_i{}^k{}_{mn} = 2 \partial_{[m} \Gamma^k_{n]i} + 2 \Gamma^l_{[n|i|} \Gamma^k_{m]l}}$$

Despite its dependendence on the non-tensor $\Gamma$, the RHS is a $\binom{1}{3}$ tensor.

<u>Review Exercise</u>: Obtain the Levi-Civita connection $\Gamma(g)^{i}_{jk}$ from the conditions

$$\nabla g = 0 = T$$

in a coordinate basis.