Lecture #2

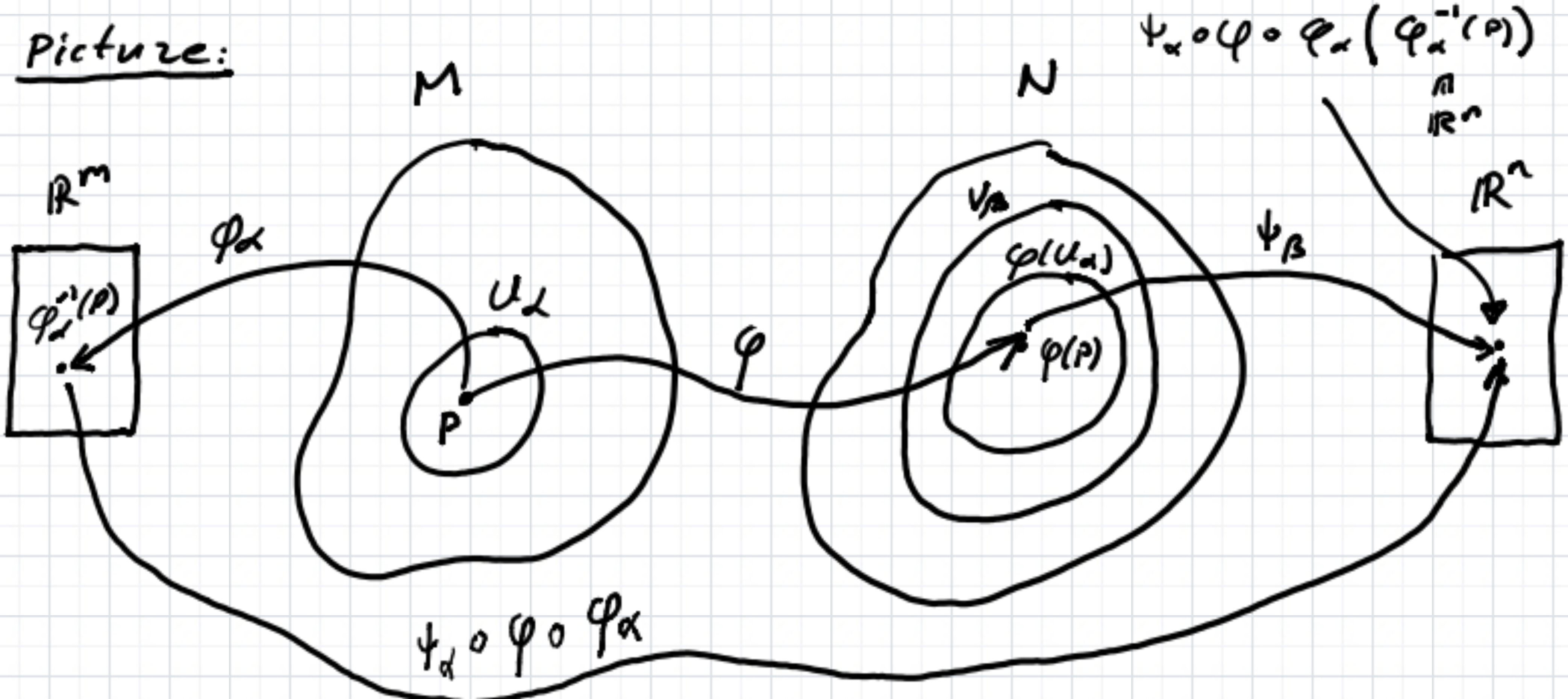
Let $\varphi : M \to N$ be a mapping between differentiable manifolds of dimension m, n

We call φ <u>differentiable</u> at $p \in M$ if for any given chart (V_β, ψ_β) at $\varphi(p)$ on N, there exists a chart $(U_\alpha, \varphi_\alpha)$ at p such that

(i) $\varphi(U_\alpha) \subset V_\beta$

(ii) $\psi_\beta \circ \varphi \circ \varphi_\alpha^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\varphi_\alpha^{-1}(p)$

<u>Picture:</u>



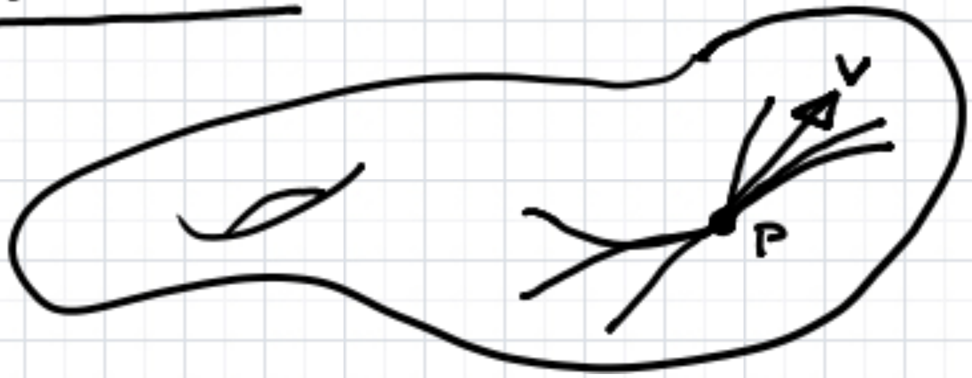
A *differentiable curve* in a manifold M is a differentiable map $\gamma : \mathbb{R} \longrightarrow M$.

A *differentiable function* is a differentiable map $f : M \longrightarrow \mathbb{R}$

TANGENT VECTORS

Two ideas

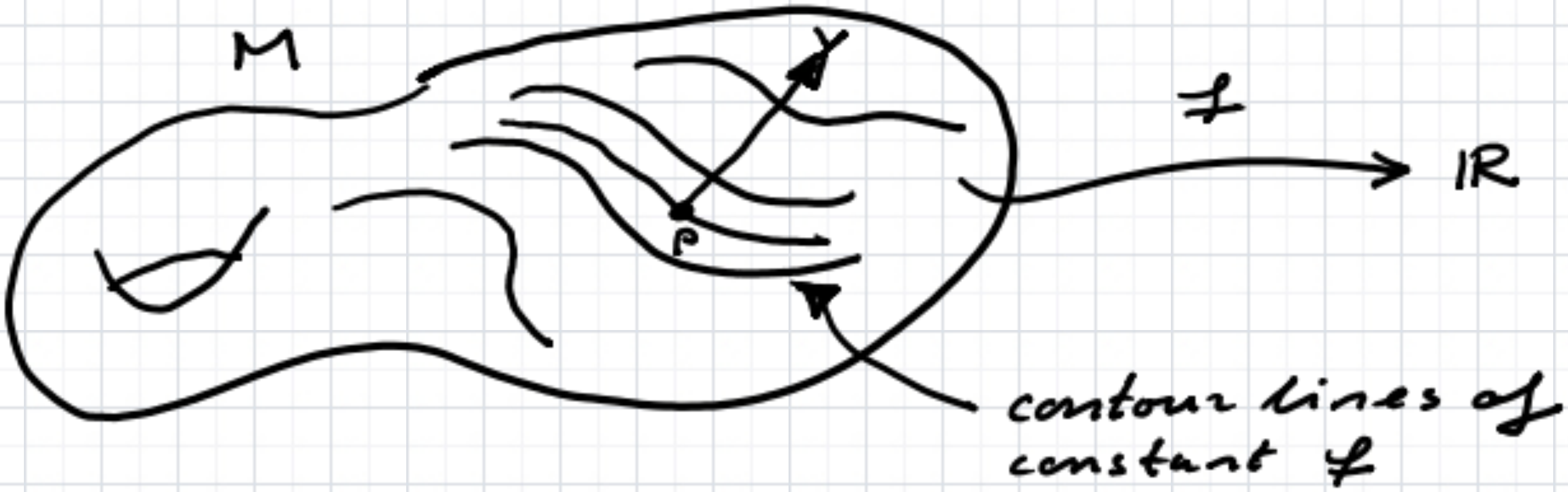
(1)



V= *Equivalence class of curves:* would expect an observer at P to be able to classify differentiable curves through P by their *speed* and *direction*

Remark: Differentiable curves are *parameterized*

②



v = <u>directional derivative</u>: would expect an observer at P to be able to determine the rate of change of any function f if you start moving in a given direction at a certain speed.

<u>DEFINITION</u> Let $\mathcal{D}(P)$ be the set of differentiable functions at P and $\gamma : \underset{\mathbb{R}}{\overset{\cap}{t}} \longmapsto M$ be a differentiable curve with $\gamma(0) = P$. The <u>tangent vector</u> $\gamma'(0) = \left.\frac{d}{dt}\right|_0$ <u>to γ at P</u> is the mapping

$$\begin{array}{rccc} \left.\frac{d}{dt}\right|_0 : & \mathcal{D}(P) & \longrightarrow & \mathbb{R} \\ & \cap & & \cap \\ & f & \longmapsto & \left.\frac{d\, f(\gamma(t))}{dt}\right|_{t=0} \end{array}$$

We are using $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ to compute derivative in the usual way.

* $\left.\frac{d}{dt}\right|_0$ defines an equivalence relation $\sim$ on curves:

  any two curves returning the same result for $(f \circ \gamma)'$ $\forall f$ are defined to be equivalent.

* Given v_p, the tangent vector to some curve γ at P, then if $f \in \mathfrak{D}(P)$, we call

$$v_p(f) \equiv \frac{d\, f \circ \gamma}{dt}(0)$$

the directional derivative of f (at P w.r.t v).

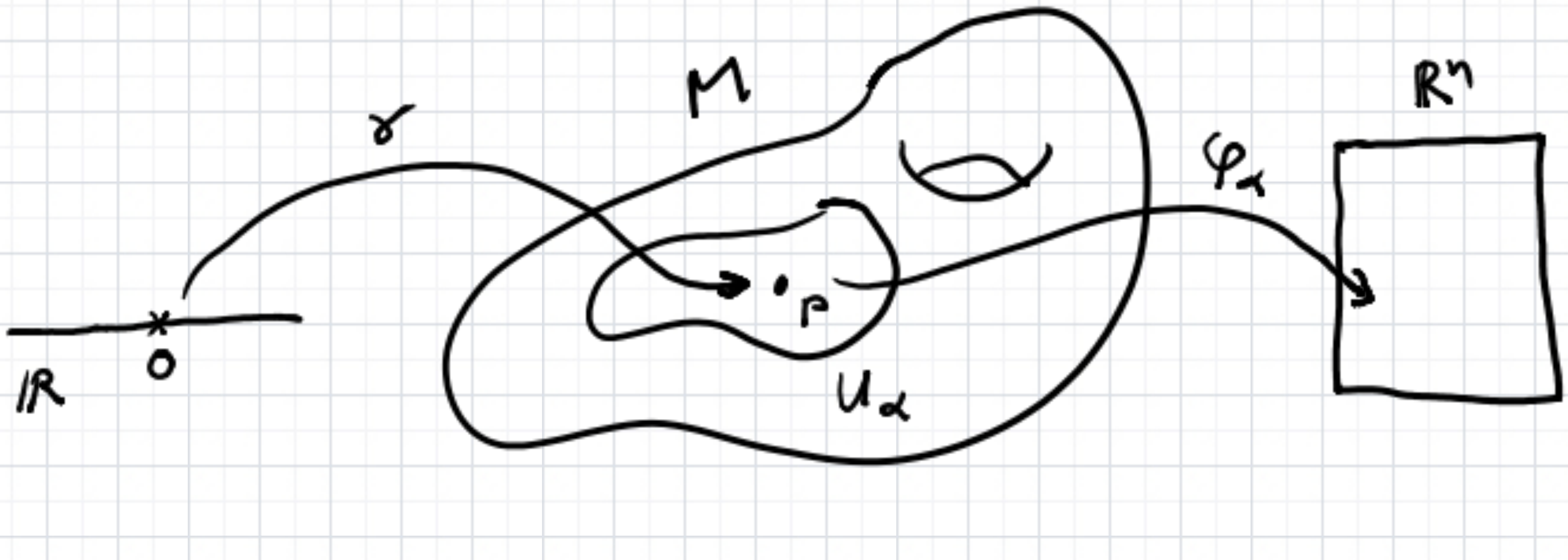
Question:

Let Γ_p be the space of differentiable curves through P. Can we find coordinates for the space $\Gamma_p/\sim$??

Let $(U_\alpha, \varphi_\alpha)$ be a chart about P where

$$M \supset U_\alpha \ni P \xmapsto[\varphi_\alpha]{} (x^1, \dots, x^n) \in \mathbb{R}^n$$

Then $t \xmapsto[\varphi_\alpha \circ \gamma]{} (x^1(t), \dots, x^n(t))$



Lets compute $v_p(f)$. Call $\bar{f} = f \circ \varphi_\alpha^{-1}$

(express f in terms of coordinate $\bar{f} = \bar{f}(x)$)

$$\left.\frac{d\, f \circ \gamma}{dt}\right|_{t=0} = \left.\frac{d\, f(\varphi_\alpha^{-1} \circ \varphi_\alpha \circ \gamma(t))}{dt}\right|_{t=0}$$

$$= \left.\frac{d\, \bar{f}(x^1(t), \ldots, x^n(t))}{dt}\right|_{t=0}$$

$$= \sum_{i=1}^{n} \frac{\partial \bar{f}}{\partial x^i} \left.\frac{dx^i}{dt}\right|_{t=0}$$

Chain rule

The coefficients $v^i \equiv \frac{dx^i(0)}{dt}$ are called the

<u>components</u> of v_p in the coordinate system x^i.

* Can always find a curve γ through p giving a set of components v^i

(try $x^i(t) = v^i t$)

* The space of vectors at p is represented by the vector space $\mathbb{R}^n \ni v^i$.

* Notice the above formula says

$$v_p(f) = \sum_i \underbrace{v^i}_{\text{vector}} \underbrace{\frac{\partial f}{\partial x^i}}_{\text{gradient}}$$

and vector $\cdot$ gradient is the $\mathbb{R}^n$ directional derivative.

Einstein summation convention

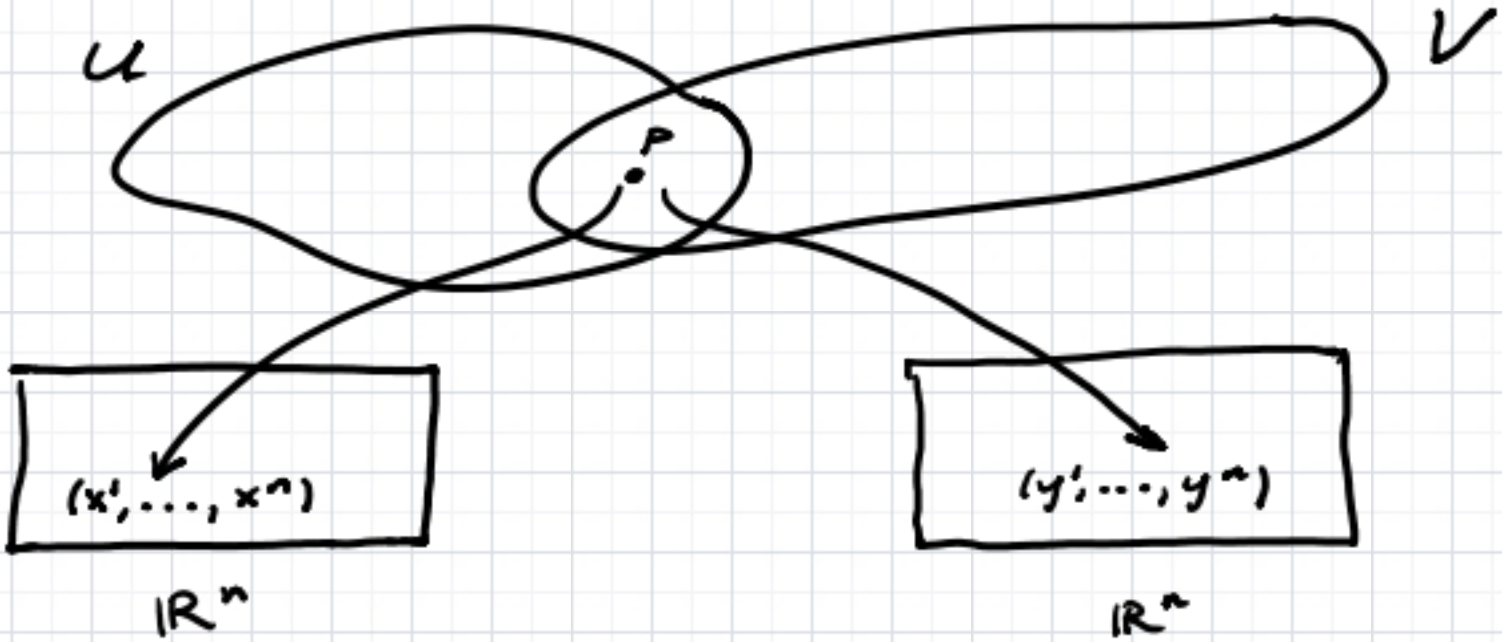
In nearly all our formulae, we can unambiguously discard the summation sign when repeated upper & lower indices appear so

$$v^i \frac{\partial}{\partial x^i} \equiv \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$$

Coordinate transformations

Just as a differentiable curve γ through p does not uniquely specify the tangent vector v_p, (but rather a representative of an equivalence class) neither does a set of components v^i:

Suppose (U, φ) & (V, ψ) are compatible charts at p:



Let

$$f : M \longrightarrow \mathbb{R}$$

and

$$\bar{f} = f \circ \varphi^{-1} \qquad \tilde{f} = f \circ \psi^{-1}$$

Observe $\bar{f} = \tilde{f} \circ \psi \circ \varphi^{-1}$

$$\Rightarrow \quad \frac{\partial \bar{f}}{\partial x^i} = \frac{\partial}{\partial x^i}\left[\tilde{f}(y(x)) \right] \qquad \text{where } y(x) = \psi \circ \varphi^{-1}(x)$$

$$= \frac{\partial \tilde{f}}{\partial y^j} \frac{\partial y^j}{\partial x^i}$$

Chain rule

Thus if v_p has components v^i in chart $(U \cap V, \varphi)$

then

$$v_p(f) = v^i \frac{\partial \bar{f}}{\partial x^i}$$

$$= v^i \frac{\partial y^j}{\partial x^i} \frac{\partial \tilde{f}}{\partial x^j}$$

New components in the chart $(U \cap V, \psi)$ are

$$\boxed{\tilde{v}^j = \frac{\partial y^j}{\partial x^i} v^i} \quad \text{——} \quad (*)$$

This rule is called the <u>transformation of a contravariant vector</u>. In many contexts the set of all components v^i with equivalence $(*)$ defined by $(*)$ is taken as the definition of a tangent vector