Lecture #20 The Riemann Tensor

We defined the Riemann tensor in terms of the operator, $u, v \in \mathfrak{X}M$

$$R(u,v) : \mathfrak{X}M \longrightarrow \mathfrak{X}M$$
$$\begin{array}{ccc} \cup & & \cup \\ w & \longmapsto & \left(\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}\right) w \end{array}$$

and defined a $\binom{3}{1}$-tensor R by

$$R(w, \alpha, u, v) = \langle \alpha, R(u,v) w \rangle .$$

Components of the Riemann tensor

$$R(\partial_i, dx^k, \partial_m, \partial_n) \equiv R_i{}^k{}_{mn} \quad , \quad \nabla_m \equiv \nabla_{\partial_m}$$

$$= \langle dx^k, \nabla_m(\nabla_n \partial_i) - (m \leftrightarrow n) \rangle = \langle dx^k, \nabla_m(\underbrace{\Gamma^j_{ni}}_{\text{function}} \partial_j) - m \leftrightarrow n \rangle$$

$$= \langle dx^k, \partial_m \Gamma^j_{ni} \partial_j + \Gamma^j_{ni} \Gamma^l_{mj} \partial_l - m \leftrightarrow n \rangle$$

$$= \boxed{\partial_m \Gamma^k_{ni} - \partial_n \Gamma^k_{mi} + \Gamma^j_{ni} \Gamma^k_{mj} - \Gamma^j_{mi} \Gamma^k_{nj} = R_i{}^k{}_{mn}}$$

Symmetries of the Riemann Tensor

From the coordinate expression, assuming vanishing torsion, it is not difficult to verify $R_{ijmn} = g_{jl} R_i{}^l{}_{mn}$ obeys

(i) $R_{ijmn} = R_{jimn}$

(ii) $R_{ijmn} = R_{mnij}$

(iii) $R_{ijmn} + R_{mijn} + R_{jmin} = 0$

(Bianchi identity of the 1st kind).

If $\dim M = n$ an antisymmetric matrix has $\frac{n(n-1)}{2}$ components so (i) & (ii) imply R_{ijmn} has

$$\frac{1}{2}\left(\frac{n(n-1)}{2}\right)\cdot\left(\frac{n(n-1)}{2}+1\right) = \frac{1}{8}\, n(n-1)(n^2-n+2)$$

mponents. Using (i) & (ii)

$R_{ijmn} + R_{mijn} + R_{jmin}$ is totally antisymmetric

This gives $\binom{n}{4}$ additional constraints and leaves

$\frac{1}{12} n^2(n^2-1)$ components.

- 0 in 1 dim
- 1 in 2 dim
- 6 in 3 dim
- 20 in 4 dim.

2^{nd} Bianchi Identity

Notice that in a coordinate basis $\{u, v, w, \dots\}$

$$R(u,v) = \nabla_u \nabla_v - \nabla_v \nabla_u$$

$$\Rightarrow \quad \nabla_w R(u,v) + \text{cyclic}$$

$$\begin{aligned} &= \nabla_w \nabla_u \nabla_v - \nabla_w \nabla_v \nabla_u \\ &+ \nabla_u \nabla_v \nabla_w - \nabla_v \nabla_u \nabla_w \\ &+ \nabla_v \nabla_w \nabla_u - \nabla_u \nabla_w \nabla_v \\ &= 0 \end{aligned}$$

[In fact true in any basis]
but there are better ways
to prove it...

i.e:

$$\boxed{\nabla_k R_{ijmn} + \nabla_i R_{jkmn} + \nabla_j R_{kimn} = 0}$$

Constant curvature manifolds

$$R(g^{-1}(\beta), \alpha, u, v) \propto \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

This says $R^{ij}{}_{mn} \propto \delta^i_m \delta^j_n - \delta^i_n \delta^j_m$

Example The sphere with its canonical metric.

Irreducible decomposition of the Riemann Tensor

Let h, k be $\binom{0}{2}$-tensors

$$(h \,\tilde{\otimes}\, k)(v_1, v_2, v_3, v_4) = h(v_1, v_3)\,k(v_2, v_4) - (1 \leftrightarrow 2) - (3 \leftrightarrow 4) + (1 \leftrightarrow 2,\ 3 \leftrightarrow 4)$$

This is sometimes called the Kulkarni Nomizu product and produces a tensor with the symmetries of Riemann. Then the $\binom{0}{4}$- Riemann-tensor R

$$\boxed{R = W + \frac{1}{n-2}\left(Ric - \frac{1}{n} s g\right) \tilde{\otimes}\, g + \frac{1}{2n(n-1)}\, s\, g \,\tilde{\otimes}\, g}$$

Where W and Ric are trace-free $\binom{0}{4}$ and $\binom{0}{2}$ tensors. s is a scalar

This formula is perhaps easier to use in components:

$$R_{ijkl} - W_{ijkl} = g_{ik} P_{jl} - g_{jk} P_{il} - g_{il} P_{jk} + g_{jl} P_{ik}$$

where

W_{ijkl} is the <u>Weyl tensor</u>.

which has the symmetries of Riemann but

$$W^i{}_{jil} = 0$$

$P_{jl} = P_{lj}$ is the <u>Schouten tensor</u>

Notice $R - W = g \circledast P$

It is built from the Ricci and scalar curvatures

$$R_{ij} = R^k{}_{ijk} \quad , \quad s = R^i{}_i$$

via

$$P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{1}{2}\frac{1}{n-1} g_{ij} s\right)$$

* The Weyl Tensor is the obstruction to conformal flatness when $n \geq 4$.

i.e. we call a manifold flat when the Riemann tensor vanishes.

If $ds^2 = \Omega^2 ds_0^2$ where ds_0^2 is a <u>flat metric</u> (the Riemann tensor for its Levi-Civita connexion vanishes) we say that (M, ds^2) is conformally flat.

* In 3-dimensions the Weyl tensor vanishes identically (and instead the "covariant curl" of the P-tensor - called the Cotton tensor - measures conformal flatness.

* There always exists a conformally equivalent metric such that the scalar curvature s is constant (Yamabe conjecture – now proven).

For a sphere it measures the inverse radius^2.

* The Ricci tensor was employed by Einstein to write equations for the space time metric. In the absence of matter, he postulated

$$Ric = 0$$

(Such manifolds are called Ricci flat and include the famous Schwarzchild solution).

Review Exercises

① Compute the difference between the Riemann tensor for a connexion with torsion and for the Levi-Civita connexion. Which symmetries of the Riemann tensor fail?

② Compute the Riemann curvature for the sphere S^2 with canonical metric

$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2.$$