Lecture #21 The Riemann Tensor

The 2nd Bianchi identity

$$0 \equiv [\nabla_u, [\nabla_v, \nabla_w]]$$

For u, v, w coordinate vector fields this gives the operator statement

$$0 = [\nabla_u, R(v,w)]$$

or in a coordinate basis

$$\boxed{\nabla_i R_j{}^k{}_{mn} + \nabla_m R_j{}^k{}_{ni} + \nabla_n R_j{}^k{}_{im} = 0}$$

Irreducible decomposition $\quad n = \dim M$

$$\boxed{R = W + \frac{1}{n-2}(Ric - sg) \otimes g - \frac{1}{2}\frac{1}{n(n-1)} sg \otimes g}$$

In components this is simpler

$$R_{ijkl} - W_{ijkl} = g_{ik}P_{jl} - g_{jk}P_{il} - g_{il}P_{jk} + g_{kl}P_{ij}$$

where W_{ijkl} are the components of the Weyl tensor.

$P_{ij} = P_{ji}$ is the Schouten tensor, a trace-adjustment of the Ricci tensor:

$$\begin{cases} R_{ij} = R^l{}_{ijl} = R_{ji}, \quad s = R_i{}^i \\ P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{1}{2}\frac{1}{n-1} s g_{ij}\right) \end{cases}$$

Notice $R - W = P \otimes g$.

This formula is perhaps easier to use in components:

$$R_{ijkl} - W_{ijkl} = g_{ik} P_{jl} - g_{jk} P_{il} - g_{il} P_{jk} + g_{jl} P_{ik}$$

where

W_{ijkl} is the <u>Weyl tensor</u>.

which has the symmetries of Riemann but

$$W^{i}{}_{jil} = 0$$

$P_{jl} = P_{lj}$ is the <u>Schouten tensor</u>

Notice $R - W = g \circledast P$

It is built from the Ricci and scalar curvatures

$$R_{ij} = R^{k}{}_{ijk} \quad , \quad s = R^{i}{}_{i}$$

via $$P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{1}{2}\frac{1}{n-1} g_{ij} s\right)$$

* The Weyl Tensor is the obstruction to conformal flatness when $n \geq 4$.

I.e. We call a manifold flat when the Riemann tensor vanishes.

If $ds^2 = \Omega^2 ds_0^2$ where ds_0^2 is a <u>flat metric</u> (the Riemann tensor for its Levi-Civita connexion vanishes) we say that (M, ds^2) is conformally flat.

* In 3-dimensions the Weyl tensor vanishes identically (and instead the "covariant curl" of the P-tensor – called the Cotton tensor – measures conformal flatness.

* There always exists a conformally equivalent metric such that the scalar curvature s is constant (Yamabe conjecture – now proven). For a sphere it measures the inverse radius^2.

* The Ricci tensor was employed by Einstein to write equations for the space time metric. In the absence of matter, he postulated

$$\mathrm{Ric} = 0$$

(such manifolds are called Ricci flat and include the famous Schwarzschild solution).

Lie Groups

G is a group with a differentiable structure such that

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (x, y) & \longmapsto & xy^{-1} \end{array}$$

This def^n ensures differentiability of y^{-1}

is differentiable. i.e. group multiplication is a differentiable mapping.

We call $T_eG \equiv \mathfrak{g}$ the Lie algebra of $\mathfrak{g}$. It is a vector space endowed with a bracket:

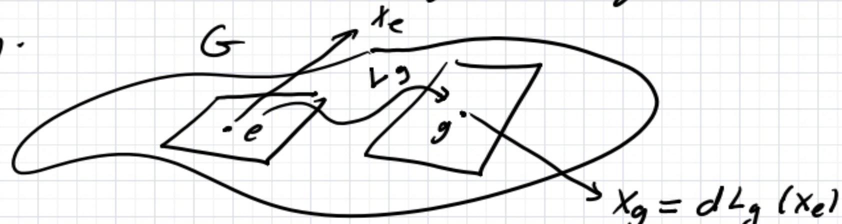
Notice, left and right multiplication

$$\left.\begin{array}{l} L_x : G \longrightarrow G \\ R_x : G \longrightarrow G \end{array}\right\} \text{via} \left\{\begin{array}{l} g \xmapsto{L_x} xg \\ g \longmapsto gx \end{array}\right.$$

are diffeomorphisms since $L_x, R_x, L_x^{-1}, R_x^{-1}$ are differentiable.

We call a metric g on G left (right) invariant if L_x (R_x) are isometries $\forall x \in G$. The same notion can be extended to arbitrary tensors.

In particular, if we know a left invariant vector field X at e (i.e. know X_e), the identity, we know it everywhere by Lie-dragging.



$$\boxed{X_g = dL_g(X_e)}$$

If X, Y are invariant vector fields, so is $[X,Y]$:

f^n around g f^n around e

$$\big(dL_g[X,Y]\big)(f) = [X,Y]_e\,(f \circ L_g)$$

vector at g f^n at g

$$= X_g\,(dL_g Y)(f) - X \leftrightarrow Y$$

$$= [X,Y]_g\,(f)$$

use $dL_g Y = Y$

i.e. $dL_g[X,Y] = [X,Y]_g$ the bracket is also invariant. So we can start $\bar{c}$ vectors in T_eM, produce left invariant vector fields and then compute their brackets.

Review Exercises

① Show $\operatorname{div}(Ric - \frac{1}{2} g s) = 0$

The tensor G is called the Einstein tensor. Look up "Mach's Principle".

Why was the tensor G important for Einstein?