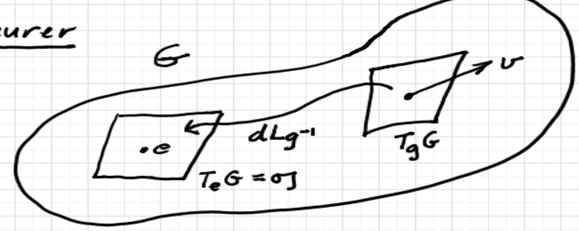
Lecture #22 LIE GROUPS

Cartan Maurer Form



Lie algebra-valued 1-form

$$\omega = dL_{g^{-1}}(\cdot) : T_gG \longrightarrow \mathfrak{g}$$

Example $G = Gl(n, \mathbb{R})$ $n\times n$ invertible matrices

$\mathfrak{g} = gl(n, \mathbb{R})$ $n \times n$ matrices with bracket = matrix commutator

The mapping

$$G \ni x \longmapsto g^A_B(x) \subset \mathbb{R}^{2n}$$

gives G a differentiable structure

If X^A_B are the components of a vector at $\mathfrak{g} = T_e Gl(n)$ the Jacobian for left multiplication L_x is just g^A_B

i.e. The left invariant vector field has coordinates $g^A_C X^C_B$ (or in a matrix notation, just gX).

So a basis for left-invariant vector fields are the operators

$$e^A_B = g^A_C \frac{\partial}{\partial g^B_C}$$

whose algebra is easy to calculate

$$[e^A_B, e^C_D] = \delta^C_B e^A_D - \delta^A_D e^C_B$$

Their duals

$$\omega^A_B\,(e^C_D) = \delta^A_D\,\delta^C_B$$

are given by

$$\omega^A_B = (g^{-1})^A_C\, dg^C_B$$

because $dg^C_B\left(\frac{\partial}{\partial g^D_E}\right) = \delta^C_D\,\delta^E_B$ so g^{-1} removes the factor g.

Often write $\omega = g^{-1}dg$, the same computation works for any Lie group written as an embedding in $GL(n,\mathbb{R})$.

Notice, left invariance is obvious,

under $g \longrightarrow hg$, $\omega \longrightarrow (hg)^{-1}d(hg)$

$$= g^{-1}h^{-1}h\,dg = \omega$$

Notice also that ω obeys a "zero curvature" condition called the Maurer-Cartan equation:

Define a "covariant derivative" or "connexion"

$$\nabla = \mathbb{1}d + \omega \qquad \text{matrix-valued operator}$$

Then $\nabla^2 = 0 \iff d\omega + \omega\wedge\omega = 0$

because

$$\nabla = d + g^{-1}(dg) = g^{-1}dg \qquad \text{as an operator}$$

$$\Rightarrow \nabla^2 = g^{-1}d\,\cancel{g}\,\cancel{g^{-1}}\,dg = g^{-1}d^2g = 0$$

Also ω can be used to form left invariant tensors, such a metric

$$ds^2 = \mathrm{tr}\,(\omega\otimes\omega) \qquad \text{also right-inv}^{\underline{t}}$$

$\Rightarrow$ Right G-action are isometries.

Lengths of Curves

Given a curve $\gamma : \mathbb{R} \to M$, the velocity vector

$$v : t \mapsto \dot{\gamma} \in T_{\gamma(t)} M$$

i.e. $v = d\gamma\left(\frac{\partial}{\partial t}\right)$ viewing $\mathbb{R}$ as a manifold.

Call the restriction γ to $I \equiv [a,b] \subset \mathbb{R}$ a segment or a path C. Define the length of C via

$$L(C) = \int_a^b dt \sqrt{g\left(\frac{dC}{dt}, \frac{dC}{dt}\right)} \equiv \int_C ds$$

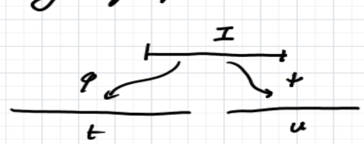
(assuming g is proper).

We can make (M,g) a metric space by defining

$$d(P,Q) = \inf_C L(C)$$

over piecewise differentiable segments C.

Notice that keeping $\mathrm{Im}(\gamma)$ unchanged but changing parameterizations



$$\frac{d}{dt} = \frac{du}{dt}\frac{d}{du}$$

but this Jacobian factor is canceled by the change of integration variable

$$dt = du \frac{dt}{du}$$

I.e. $L(\cdot)$ is a functional of unparameterized paths.

For many applications the square root is inconvenient.

It is more convenient to study the energy integral

$$E(C) = \frac{1}{2}\int_a^b dt\, g(\dot{C}, \dot{C})$$

but $E(\cdot)$ is a functional of parameterized paths. Both $E(\cdot)$ and $L(\cdot)$ are extremized by geodesics. For this we need to study <u>variations</u> and action principles.
In fact, there is even a third action principle which gives $\mathbb{R}$ also a metric.

<u>Exercise:</u> Show $\omega^A_{\ B} = (g^{-1})^A_{\ C}\, dg^C_{\ B}$ is the Cartan Maurer form for $Gl(n)$.

Review Exercises

① Show $\operatorname{div}(Ric - \frac{1}{2} g s) = 0$

The tensor G is called the Einstein tensor. Look up "Mach's Principle".

Why was the tensor G important for Einstein?