Lecture #24 VARIATIONS

A variation of a path

$$c : [a,b] \longrightarrow M$$

is a 1-parameter family of paths

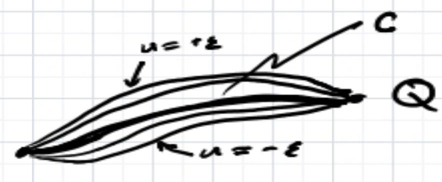
$$c_u : [a,b] \longrightarrow M$$

such that

i) $c_0 = c$

ii) $c_u(t)$ is a differentiable function of u and t for $a \le t \le b$ and u in some neighborhood $\varepsilon \geq |u|$.

iii) endpoints: $c_u(a) = c(a)$ $\quad \forall\ |u| \le \varepsilon$
$c_u(b) = c(b)$



We say the functional $S : \Omega(P,Q) \longrightarrow \mathbb{R}$ is stationary at $c \in \Omega(P,Q)$, the set of differentiable paths joining $P, Q \in M$

if

$$\frac{d}{du} S(c_u)\Big|_{u} = 0$$

$\forall$ variations c_u of c.

An example of $S : \Omega(p,q) \to \mathbb{R}$ is the energy

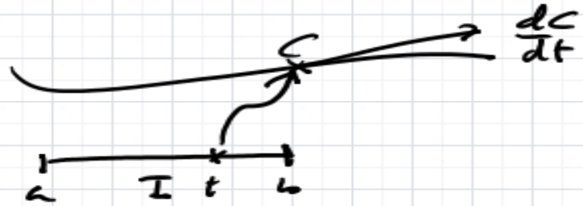
$$E(C) = \frac{1}{2} \int_C dt \, g(\dot{c}, \dot{c})$$

Generically, a functional whose extrema determine the dynamics of a system is called an <u>action</u>, and hence the term "<u>principle of least action</u>".

<u>Definition</u>: The map

$$\tilde{C} : t \in I \longmapsto \Big(\underbrace{c(t)}_{\text{position}} , \underbrace{\frac{dc}{dt}}_{\text{velocity}} , \underbrace{t}_{\substack{\text{parameterization} \\ = \text{time}}} \Big) \in TM \times \mathbb{R}$$

is called the <u>lift</u> of the curve C.



Often study $S(C) = \int_a^b dt \, L \circ \tilde{C}(t)$

where $L : TM \times \mathbb{R} \longrightarrow \mathbb{R}$

is called the <u>Lagrangian</u>.

EXAMPLE For the energy $E(c)$,

$$L : (c, \dot{c}, t) \longmapsto \tfrac{1}{2} g(\dot{c}, \dot{c})$$

* t-independent, quadratic in $\dot{c}$, possibly very non-trivial c-dependence via g.

We need to compute $\frac{d}{du} S(c_u)\Big|_{u=0}$,

first

$$\frac{d}{du} L(\tilde{c}_u) = dL \circ \frac{\partial \tilde{c}_u}{\partial u}$$

(here $\tilde{c}_u$ at fixed t is a curve parameterized by u)

vector field in $T(TM \times \mathbb{R})$

So S is ~~stationary iff~~

$$\int_a^b dt \left[dL \circ \frac{\partial \tilde{c}_u}{\partial u} \right]_{u=0} = 0$$

W.l.o.g. assume c is contained in some chart φ so

$$L \circ \varphi^{-1} = L(x^i, v^i, t) \quad , \quad \tilde{c}^i = (\varphi \circ c)^i$$

positions

coordinates of velocity vector

$$\Rightarrow \left[dL \circ \frac{\partial \tilde{c}_u}{\partial u} \right] = \left\{ \frac{\partial L}{\partial x^i} \frac{\partial c_u^i}{\partial u} + \frac{\partial L}{\partial v^i} \frac{\partial^2 \tilde{c}_u^i}{\partial u \, \partial t} \right\}_{u=0}$$

Observe

$$\frac{\partial L}{\partial v^i}\frac{\partial^2 \tilde{c}^i}{\partial u\, \partial t} = \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\frac{\partial \tilde{c}^i}{\partial u}\right) - \left(\frac{d}{dt}\frac{\partial L}{\partial v^i}\right)\frac{\partial \tilde{c}_u^i}{\partial t}$$

can discard this
term because
$c_u(t=a,b)$ is constant
(fixed endpoints +
fundamental theorem
of calculus)

Call $\quad \mathcal{G}_i \equiv \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial v^i}$ (components of a 1-form in $T^*(TM)$)

Then

$$\frac{d}{du} S(c_u)\Big|_{u=0} = \int_a^b dt \left\langle \underset{T^*(TM)}{\underset{\cap}{\mathcal{G}}},\ \underset{T(TM)}{\underset{\cap}{\frac{\partial c_u}{\partial u}}} \right\rangle_{u=0}$$

But this integral must vanish for any variation c_u so

$$\boxed{0 = \mathcal{G}_i = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial v^i}}$$ Euler Lagrange Equations

__Example__ The Energy

$$L(x^i, v^i) = \frac{1}{2} g(v, v) = \frac{1}{2} g_{ij} v^i v^j$$

$$\frac{\partial L}{\partial x^i} = \frac{1}{2} \partial_i g_{jk} v^j v^k$$

$$\frac{\partial L}{\partial v^i} = g_{ij} v^j$$

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} = g_{ij} \dot{v}^j + \dot{x}^k \partial_k g_{ij} v^j = g_{ij} \dot{v}^j + \partial_k g_{ij} v^k v^j$$

so the Euler-Lagrange equations say

$$0 = g_{ij} \dot{v}^j + \underbrace{\frac{1}{2}(\partial_k g_{ji} + \partial_j v_{ki} - \partial_i g_{jk})}_{\Gamma_{ijk}} v^j v^k$$

$$\Rightarrow \boxed{0 = \dot{v}^l + \Gamma^l_{jk} v^k v^j = \frac{\nabla v^l}{dt}} \quad \text{\underline{Geodesic Equation}}$$

Local __requirement to minimize length__

Other action principles

The energy integral was parameterization dependant. Can we repair this?

The energy integral pulls g back to I via dC. Give I its own intrinsic metric γ which we can always write

$$\gamma = de \otimes de$$

where the "einbein" e is a 1-form on I.

$$e = \bar{e}\, dt$$

and under changes of coordinates $t \to t'(t)$

$$\bar{e}' = \frac{dt'}{dt}\, \bar{e}$$

Thus $\int_I e = \int_I \bar{e}\, dt$ $\underline{\underline{\text{is}}}$ reparameterization independent.

(Example of general fact – can integrate top form over a manifold.)

But $\int \frac{dt}{\bar{e}}\, g(\dot{c},\dot{c})$ is also reparameterization invariant.

Hence study

$$\boxed{S = \frac{1}{2}\int \frac{dt}{\bar{e}}\, g(\dot{c},\dot{c}) + \frac{m^2}{2}\int dt\, \bar{e}} \quad (*)$$

varying $c \to c_u$ still get geodesic equation, but varying $\bar{e}$ we obtain

$$\frac{\delta S}{\delta \bar{e}} = -\frac{1}{\bar{e}^2}\, g(\dot{c},\dot{c}) + m^2 = 0$$ Euler Lagrange equation for $\bar{e}$.

$\Rightarrow \bar{e} = \frac{1}{m}\sqrt{g(\dot{c},\dot{c})}$ plugging back in S gives

$\Rightarrow S = \frac{1}{m}\int dt\, \sqrt{g(\dot{c},\dot{c})} = L(c)$ the **length!**

Special case $m=0$, Euler Lagrange eq$^{\underline{n}}$ for $\bar{e}$ says

$g(\dot{c},\dot{c}) = 0 \rightsquigarrow$ non-sense for g proper

For Lorentzian manifolds this is the light like condition!

Finally note that since $(*)$ is reparameterization invariant, we can choose coordinates on I $t'(t)$ such that $\bar{e}' = \underline{\text{constant}} = 1$ (say)

$\Rightarrow S = \frac{1}{2}\int dt\, g(\dot{c},\dot{c}) + \underline{\text{constant}}$, the **energy** integral.

Exercise: Show that the geodesic implies

$$\frac{d}{dt} g(v,v) = 0 .$$

ie. the speed $s^2 = |v|^2$ is constant.

Review Exercises

① Show $\mathrm{div}\left(Ric - \frac{1}{2} g\, s\right) = 0$

The tensor G is called the Einstein tensor. Look up "Mach's Principle".

Why was the tensor G important for Einstein?