Lecture #25 ACTION PRINCIPLES

$$S(C) = \int dt\, L \circ \tilde{C}(t) \qquad \underline{\text{action}}$$

where

$$t \xrightarrow{\tilde{C}} (c(t), \dot{c}(t), t) \quad \text{is lift of } C$$

$$L : TM \otimes \mathbb{R} \longrightarrow \mathbb{R} \quad \text{is the } \underline{\text{Lagrangian}}.$$

Varying $S(C)$ we found $\dfrac{\delta S}{\delta C} = 0$ when

$$\boxed{\mathcal{G}_i = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0}$$

time indep$^{\underline{t}}$
Euler-Lagrange eq$^{\underline{n}}$s.

components of a 1-form on TM.

Example The Energy

$$L(x^i, v^i) = \tfrac{1}{2} g(v,v) = \tfrac{1}{2} g_{ij} v^i v^j$$

$$\frac{\partial L}{\partial x^i} = \tfrac{1}{2} \partial_i g_{jk} v^j v^k$$

$$\frac{\partial L}{\partial v^i} = g_{ij} v^j$$

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} = g_{ij} \dot{v}^j + \dot{x}^k \partial_k g_{ij} v^j = g_{ij} \dot{v}^j + \partial_k g_{ij} v^k v^j$$

so the Euler-Lagrange equations say

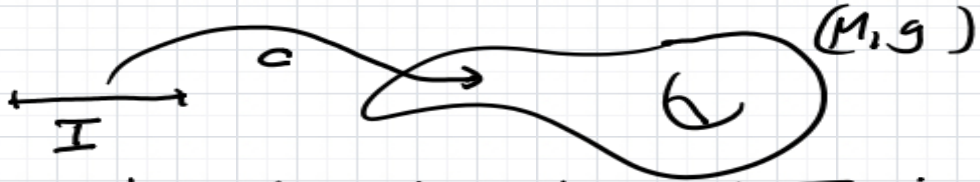
$$0 = g_{ij} \dot{v}^j + \underbrace{\tfrac{1}{2}\left(\partial_k g_{ji} + \partial_j v_{ki} - \partial_i g_{jk}\right)}_{\Gamma_{ijk}} v^j v^k$$

$$\Rightarrow \boxed{0 = \dot{v}^l + \Gamma^l_{jk} v^k v^j = \frac{\nabla v^l}{dt}}$$ Geodesic Equation

Local requirement to minimize length

Other action principles

The energy integral was parameterization dependent. Can we repair this?



The energy integral pulls g back to I via dC.
Give I its own intrinsic metric γ which we can always write

$$\gamma = de \otimes de$$

where the "einbein" e is a 1-form on I.

$$e = \bar{e}\, dt$$

and under changes of coordinates $t \to t'(t)$

$$\bar{e}' = \frac{dt'}{dt}\, \bar{e}$$

Thus $\int_I e = \int_I \bar{e}\, dt$ $\underline{\underline{\text{is}}}$ reparameterization independent.

(Example of general fact – can integrate top form over a manifold.)

But $\int \frac{dt}{\bar{e}}\, g(\dot{c},\dot{c})$ is also reparameterization invariant.

Hence study

$$\boxed{S = \frac{1}{2}\int \frac{dt}{\bar{e}}\, g(\dot{c},\dot{c}) + \frac{m^2}{2}\int dt\, \bar{e}} \quad \text{— } (*)$$

varying $c \to c_u$ still get geodesic equation, but varying $\bar{e}$ we obtain

$$\frac{\delta S}{\delta \bar{e}} = -\frac{1}{\bar{e}^2}\, g(\dot{c},\dot{c}) + m^2 = 0 \qquad \text{Euler Lagrange equation for } \bar{e}.$$

$\Rightarrow \bar{e} = \frac{1}{m}\sqrt{g(\dot{c},\dot{c})}$ plugging back in S gives

$\Rightarrow S = \frac{1}{m}\int dt \sqrt{g(\dot{c},\dot{c})} = L(c)$ the length!

Special case $m = 0$, Euler Lagrange eq^{n} for $\bar{e}$ says

$g(\dot{c},\dot{c}) = 0 \rightsquigarrow$ non-sense for g proper

For Lorentzian manifolds this is the light like condition!

Finally note that since $(*)$ is reparameterization invariant, we can choose coordinates on I $t'(t)$ such that $\bar{e}' = \underline{\text{constant}} = 1$ (say)

$\Rightarrow S = \frac{1}{2}\int dt\, g(\dot{c},\dot{c}) + \underline{\text{constant}}$, the energy integral.

Lie algebra cohomology

(Fuchs, Cohomology of Infinite dimensional Lie Algebras)
arXiv: 0906.4814

Let $\mathcal{M}$ be some module, typically sections of some bundle over M. Examples might include $\mathfrak{X}M$, ΛM, ...

Suppose the (super) Lie algebra

$$\mathfrak{g} = \{g_A\} \quad , \quad [g_A, g_B\} = f_{AB}^{\ \ C} g_C$$

is represented by a set of operators

$$g_A : \mathcal{M} \longrightarrow \mathcal{M}$$

where

$$\left(g_A g_B - (-)^{AB} g_B g_A - f_{AB}^{\ \ C} g_C\right)\omega = 0$$

for $\omega \in \mathcal{F}M$. Examples include $g_A = \mathcal{L}_{\xi_A}$ for vector fields on $\mathfrak{X}M$ or the exterior derivative d on ΛM

$$d\,d\,\omega = 0 \qquad \{g_A\} = \{d\}, \ f_{BC}^{\ \ A} = 0$$

Consider the graded polynomial algebra

$$k[[b_A, c^A]] / \mathcal{H}$$

where $\mathcal{H}$ are the relations

$$\left.\begin{aligned} [b_A, b_B\} &= 0 \\ [b_A, c^B\} &= \delta_A^B \\ [c^A, c^B\} &= 0 \end{aligned}\right\}$$

b, c have opposite grading to the "charges" g_A

Claim the operator

$$Q_g = c^A c^B f_{BA}^C b_C$$

is nilpotent $Q_g^2 = 0$.

You can check this using the (super) Jacobi identity for g and the relations above.

Alternatively the relationship

$$g_A\, g_B - (-)^{AB} g_B\, g_A - f_{AB}^{C}\, g_C = 0$$

naturally implies an exact sequence:

Writing out any of these relationships:

$$g_1 g_2 - g_2 g_1 = g_3 \qquad \text{(say)}$$

says

$$\underbrace{\begin{pmatrix} -g_2 & g_1 & -1 \end{pmatrix}}_{Q^{(2)}} \underbrace{\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}}_{Q^{(1)}} = 0$$

$$0 \longrightarrow \mathcal{M} \xrightarrow{Q^{(1)}} \mathcal{M}^{\otimes \dim \mathfrak{g}} \xrightarrow{Q^{(2)}} \mathcal{M}^{\otimes \left(\begin{smallmatrix} \dim \mathfrak{g} \\ 2 \end{smallmatrix}\right]} \longrightarrow \cdots$$

"The superchoose 2 function"

This complex is generated by

$$Q = c^A g_A + \tfrac{1}{2} c^A c^B f_{AB}^{C}\, b_C$$

Maxwell's and linearized Einstein's Equations from Lie Algebra Cohomology

The space of sections of ΛM – differential forms ΩM

or $\odot TM$ – totally symmetric tensors SM

are endowed with some use ful operators:

Give them a unified description by viewing

differential & "symmetric" form

as (polynomial) functions of anticommuting/commuting differentials dx^μ

forms $\quad dx^\mu dx^\nu = -dx^\nu dx^\mu$ (the wedge product)

symmetric forms $\quad dx^\mu dx^\nu = +dx^\nu dx^\mu$

Degree/Number operator

$$N = dx^\mu \frac{\partial}{\partial(dx^\mu)}$$ (No metric required.)

Gradient $\quad dx^\mu \nabla_\mu$

For forms this is the exterior derivative d

For symmetric forms call this operator grad

$$\begin{aligned} \mathrm{grad} : S^k M &\longrightarrow S^{k+1} M \\ \varphi_{\mu_1 \cdots \mu_s} &\longmapsto \nabla_{(\mu_1} \varphi_{\mu_2 \cdots \mu_{s+1})} \end{aligned}$$

Notice $\quad N\, d = d\,(N+1) \qquad \{d, d\} = \frac{1}{2} d^2 = 0$

$$N\, \mathrm{grad} = \mathrm{grad}\,(N+1)$$

Divergence $\quad \dfrac{\partial}{\partial(dx^\mu)} \nabla^\mu$

For forms this operator is called δ, the codifferential

$$\delta : \Omega^k M \longrightarrow \Omega^{k-1} M$$

The codifferential obeys

$$N\delta = \delta N - 1 \qquad \delta^2 = 1$$

$$d\delta + \delta d = \Delta$$

where Δ is the "form Laplacian". It is a curvature modification of the "Bochner Laplacian" $g^{\mu\nu}\nabla_\mu\nabla_\nu$. The form Laplacian is central.

For symmetric forms, the above operator is the divergence

$$\begin{array}{rcl} \text{div}: S^k M & \longrightarrow & S^{k-1}M \\ \psi & & \\ \varphi_{\mu_1\cdots\mu_k} & \longmapsto & k\,\nabla^\mu \varphi_{\mu\mu_2\cdots\mu_k} \end{array}$$

again there is a nice algebra (due to Lichnerowicz

$$N\,\text{div} = \text{div}(N-1)$$

$$\text{div grad} - \text{grad div} = \overset{\text{Bochner}}{\Delta} + \underbrace{\tfrac{1}{4}R^{\#\#}}_{\text{curvature modification}}$$

But the right hand side is generically <u>not</u> central.

For symmetric forms there are two other new operators

$$g = g_{\mu\nu}\, dx^\mu dx^\nu : S^k M \longrightarrow S^{k+2} M$$

$$g_{\mu_1 \cdots \mu_k} \longmapsto g_{(\mu_1\mu_2} g_{\mu_3 \cdots \mu_{k+2})}$$

$$tr = g^{\mu\nu} \frac{\partial}{\partial(dx^\mu)} \frac{\partial}{\partial(dx^\nu)} : S^k M \longrightarrow S^{k-2} M$$

$$g_{\mu_1 \cdots \mu_k} \longmapsto k(k-1)\, g^{\mu}{}_{\mu\mu_3 \cdots \mu_k}$$

The operators g, N, tr obey an $sl(2)$ Lie algebra

$$[N, g] = 2g$$

$$[N, tr] = -2tr$$

$$[tr, g] = 4N + 2\dim M$$

Next time study Lie algebra cohomology of these two algebras.

<u>Review</u> ① We wrote an action principle to minimize lengths of curves.

Study its generalization to

(i) Areas
(ii) Volumes.

② Look up Hodge duality. In particular find out how to map k-forms to $(\dim M - k)$-forms when M has a metric ds^2.