Lecture 26 Lie Algebra Cohomology

Start with two representations of a Lie superalgebra $\mathfrak{g}$

$$\begin{cases} g_A : \mathcal{M} \longrightarrow \mathcal{M} \\ g_A^{ghost} : k[[c^A]] \longrightarrow k[[c^A]] \end{cases}$$

where $\mathcal{M}$ is some module/vector space (think sections of some tensor bundle over M), and

$$g_A^{ghost} = f_{BA}^{C} c^B \frac{\partial}{\partial c^C}$$

In particular g_A and g_A^{ghost} can both act on $\mathcal{M}[[c^A]]$ and

$$Q = c^A (g_A + g_A^{ghost}) : \mathcal{M}[[c^A]] \to \mathcal{M}[[c^A]]$$

squares to zero.

The cohomology $H^\bullet(\mathfrak{g}, \mathcal{M})$ of the complex

$$\xrightarrow{Q} \mathcal{M}[[c^A]] \xrightarrow{Q}$$

is called the Lie algebra cohomology of $\mathfrak{g}$ with respect to $\mathcal{M}$. The differential Q is called the Chevalley–Eilenberg differential.

Consider two cases

$$\mathcal{M} = \begin{cases} \text{symmetric forms } \Gamma(\odot TM) \\ \text{differential forms } \Omega M \end{cases}$$

Maxwell's and linearized Einstein's Equations from Lie Algebra Cohomology

The space of sections of ΛM – differential forms ΩM

or $\odot TM$ – totally symmetric tensors SM

are endowed with some useful operators:

Give them a unified description by viewing differential & "symmetric" form as (polynomial) functions of anticommuting/commuting differentials dx^μ

forms $\quad dx^\mu dx^\nu = - dx^\nu dx^\mu$ {the wedge product}

symmetric forms $\quad dx^\mu dx^\nu = + dx^\nu dx^\mu$

Degree / Number operator

$$N = dx^\mu \frac{\partial}{\partial(dx^\mu)}$$ (No metric required.)

Gradient $\quad dx^\mu \nabla_\mu$

For forms this is the exterior derivative d

For symmetric forms call this operator grad

$$\text{grad} : S^k M \longmapsto S^{k+1} M$$
$$\psi \qquad\qquad \psi$$
$$\varphi_{\mu_1 \cdots \mu_s} \longmapsto \nabla_{(\mu_1} \varphi_{\mu_2 \cdots \mu_{s+1})}$$

Notice $\quad N\, d = d\,(N+1) \qquad \{d, d\} = \frac{1}{2} d^2 = 0$

$$N\, \text{grad} = \text{grad}\,(N+1)$$

Divergence $\quad \frac{\partial}{\partial(dx^\mu)} \nabla^\mu$

For forms this operator is called δ, the codifferential

$$\delta : \Omega^k M \longrightarrow \Omega^{k-1} M$$

The <u>codifferential</u> obeys

$$N\delta = \delta N - 1 \qquad \delta^2 = 1$$

$$d\delta + \delta d = \Delta$$

where Δ is the "form Laplacian". It is a curvature modification of the "Bochner Laplacian" $g^{\mu\nu}\nabla_\mu\nabla_\nu$. The form Laplacian is central.

For symmetric forms, the above operator is the divergence

$$\mathrm{div}: S^kM \longrightarrow S^{k-1}M$$

$$\varphi_{\mu_1\cdots\mu_k} \longmapsto k\,\nabla^\mu \varphi_{\mu\mu_2\cdots\mu_k}$$

again there is a nice algebra (due to <u>Lichnerowicz</u>

$$N\,\mathrm{div} = \mathrm{div}(N-1)$$

$$\mathrm{div}\,\mathrm{grad} - \mathrm{grad}\,\mathrm{div} = \overset{\text{Bochner}}{\Delta} + \underbrace{\tfrac{1}{4}R^{\#\#}}_{\text{curvature modification}}$$

But the right hand side is generically <u>not</u> central.

For symmetric forms there are two other new operators

$$g = g_{\mu\nu}\,dx^\mu dx^\nu : S^kM \longrightarrow S^{k+2}M$$

$$\varphi_{\mu_1\cdots\mu_k} \longmapsto g_{(\mu_1\mu_2}\varphi_{\mu_3\cdots\mu_{k+2})}$$

$$tr = g^{\mu\nu}\frac{\partial}{\partial(dx^\mu)}\frac{\partial}{\partial(dx^\nu)} : S^kM \longrightarrow S^{k-2}M$$

$$\varphi_{\mu_1\cdots\mu_k} \longmapsto k(k-1)\,\varphi^\mu{}_{\mu\mu_3\cdots\mu_k}$$

The operators g, N, tr obey an $sl(2)$ Lie algebra

$$[N,g] = 2g$$

$$[N,tr] = -2tr$$

$$[tr,g] = 4N + 2\dim M$$

Take $\mathfrak{g} = \{ \mathrm{grad}, \Delta, \mathrm{div}, \mathrm{tr} \}$

$[\mathrm{tr}, \mathrm{grad}] = 2\,\mathrm{div}, \quad [\mathrm{div}, \mathrm{grad}] = \Delta$

$$\Rightarrow \underbrace{\begin{pmatrix} \mathrm{tr} & 0 & \boxed{-2} & -\mathrm{grad} \\ 0 & \mathrm{tr} & 0 & -\Delta \\ 0 & 0 & \mathrm{tr} & -\mathrm{div} \\ \mathrm{div} & \boxed{-1} & -\mathrm{grad} & 0 \\ 0 & \mathrm{div} & -\Delta & 0 \\ \Delta & -\mathrm{grad} & 0 & 0 \end{pmatrix}}_{Q^{(1)}} \underbrace{\begin{pmatrix} \mathrm{grad} \\ \Delta \\ \mathrm{div} \\ \mathrm{tr} \end{pmatrix}}_{Q^{(0)}} = 0$$

$$\Rightarrow \underbrace{Q^{(1)} \begin{pmatrix} \varphi_{\mathrm{grad}} \\ \varphi_{\Delta} \\ \varphi_{\mathrm{div}} \\ \varphi_{\mathrm{tr}} \end{pmatrix} = 0}_{\text{closed}}, \quad \underbrace{\begin{pmatrix} \varphi_{\mathrm{grad}} \\ \varphi_{\Delta} \\ \varphi_{\mathrm{div}} \\ \varphi_{\mathrm{tr}} \end{pmatrix} \sim \begin{pmatrix} \mathrm{grad} \\ \Delta \\ \mathrm{div} \\ \mathrm{tr} \end{pmatrix} \alpha}_{\text{modulo exact}}$$

The appearance of structure constants is good — it allows us to solve for some "fields"

$$\begin{cases} \mathrm{tr}\,\varphi_{\mathrm{grad}} - \mathrm{grad}\,\varphi_{\mathrm{tr}} = 2\varphi_{\mathrm{div}} \\ \mathrm{div}\,\varphi_{\mathrm{grad}} - \mathrm{grad}\,\varphi_{\mathrm{div}} = \varphi_{\Delta} \end{cases}$$

Exactness is also useful, for example

$$\varphi_{\mathrm{tr}} \sim \varphi_{\mathrm{tr}} + \mathrm{tr}\,\alpha$$

is algebraic, use it to choose $\boxed{\varphi_{\mathrm{tr}} = 0}$.
Only $\alpha = \alpha_0 \in \ker \mathrm{tr}$ remains.

$$\Rightarrow \quad \varphi_{\mathrm{div}} = \tfrac{1}{2}\,\mathrm{tr}\,\varphi_{\mathrm{grad}}$$

$$\varphi_{\Delta} = \left(\mathrm{div} - \tfrac{1}{2}\,\mathrm{grad}\,\mathrm{tr}\right)\varphi_{\mathrm{grad}}$$

Hence only φ_{grad} is indep$^{\underline{t}}$.

But

$$\mathrm{grad}\,\varphi_\Delta - \Delta\,\varphi_{\mathrm{grad}} = 0$$

$$\Rightarrow \Big(\underbrace{\Delta + \tfrac{1}{2}\mathrm{grad}^2\mathrm{tr}}_{-\,\mathrm{grad\,div}}\Big)\varphi_{\mathrm{grad}} = 0$$

And

$$\mathrm{tr}\,\varphi_{\mathrm{div}} - \mathrm{div}\,\varphi_{\mathrm{tr}} = 0$$

$$\Rightarrow \tfrac{1}{2}\mathrm{tr}^2\varphi_{\mathrm{grad}} = 0$$

Hence
$$\begin{cases} ① & (\Delta - \mathrm{grad\,div} + \tfrac{1}{2}\mathrm{grad}^2\mathrm{tr})\varphi_{\mathrm{grad}} = 0 \\ ② & \mathrm{tr}^2\varphi_{\mathrm{grad}} = 0 \end{cases}$$

with "gauge invariance" $\varphi_{\mathrm{grad}} \sim \varphi_{\mathrm{grad}} + \mathrm{grad}\,\alpha_0$

For $\varphi_{\mathrm{grad}} = h_{\mu\nu}\,dx^\mu dx^\nu \equiv h$

"metric fluctuations $ds^2 = ds^2_{\mathrm{flat}} + h$"

① is the Linearized Einstein Equation

② are linearized diffeomorphisms.

Exercises ① Show $Q = c^A(g_A + g_A^{\mathrm{ghost}})$
squares to zero.

② Study the Lie Algebra cohomology of the operator

$$Q = c\Delta + \frac{\partial}{\partial p}d + z\delta - z\frac{\partial}{\partial p}\frac{\partial}{\partial c}$$

acting as $\Omega M[[z,p]]$

where

$$d\delta + \delta d = \Delta$$