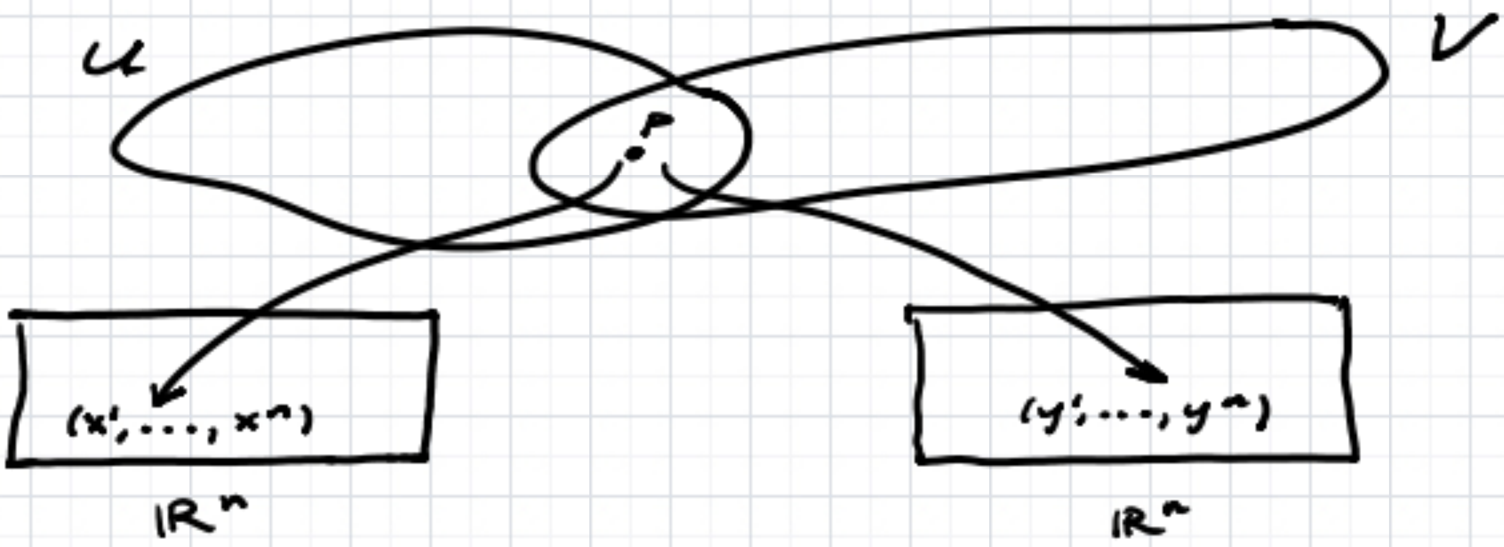
Lecture #3

Coordinate transformations

Just as a differentiable curve γ through p does not uniquely specify the tangent vector v_p, (but rather a representative of an equivalence class) neither does a set of components v^i:

Suppose (U,φ) & (V,ψ) are compatible charts at p:



Let

$$f : M \longrightarrow \mathbb{R}$$

and

$$\bar{f} = f \circ \varphi^{-1} \qquad \tilde{f} = f \circ \psi^{-1}$$

Observe $\bar{f} = \tilde{f} \circ \psi \circ \varphi^{-1}$

$$\Rightarrow \frac{\partial \bar{f}}{\partial x^i} = \frac{\partial}{\partial x^i}\left[\tilde{f}(y(x)) \right] \quad \text{where } y(x) = \psi \circ \varphi^{-1}(x)$$

$$= \frac{\partial \tilde{f}}{\partial y^j} \frac{\partial y^j}{\partial x^i}$$

Chain rule

Thus if v_p has components v^i in chart $(U \cap V, \varphi)$

then
$$v_p(f) = v^i \frac{\partial \bar{f}}{\partial x^i}$$

$$= v^i \frac{\partial y^j}{\partial x^i} \frac{\partial \tilde{f}}{\partial x^j}$$

New components in the chart $(U \cap V, \psi)$ are

$$\boxed{\tilde{v}^j = \frac{\partial y^j}{\partial x^i} v^i} \quad \text{——} \quad (*)$$

This rule is called the <u>transformation of a contravariant vector</u>. In many contexts the set of all components v^i with equivalence $(*)$ defined by $(*)$ is taken as the definition of a tangent vector

We saw tangent vectors were

1. equivalence classes of curves

2. directional derivative

3. contravariant components

Another viewpoint

Let A be an algebra over a field F

We call a linear operator D a <u>derivation</u> of A if

$$D(xy) = D(x)y + xD(y) \quad \forall x, y \in A$$

We say that differentiable functions

$$f_1, f_2 : M \longrightarrow R$$

have the same <u>germ</u> at $P \in M$ if $\exists$ an open neighborhood of P on which $f_1 = f_2$

(Only need <u>local</u> knowledge of f to compute derivatives)

We can use "having the same germ" to define an equivalence relation on differentiable functions.

* The equivalence class of $f: M \to \mathbb{R}$ is called the germ of f at P
* The set of all germs forms an algebra because we can multiply & add functions.

We can now define <u>tangent vectors</u> as derivations on the algebra of germs of differentiable functions at P.

TANGENT VECTOR SPACE

<u>Summary</u>: A tangent vector v_p is a linear functional $\mathcal{D}(P) \to \mathbb{R}$ subject to

(i) $v_p(\alpha f + \beta g) = \alpha v_p(f) + \beta v_p(g)$

$$f, g \in \mathcal{D}(P); \alpha, \beta \in \mathbb{R}$$

(ii) $v_p(fg) = v_p(f) g + f v_p(g)$

The space $T_p(M)$ of tangent vectors to M at P is called the <u>tangent vector space</u>.

A manifold equipped $\bar{c}$ tangent spaces at all $P \in M$ is called the <u>tangent bundle</u>.

The natural basis for $T_p M$

Given a coordinate system x^i we can construct differentiable curves γ_j

only j^{th} slot non-vanishing

$$\varphi_\alpha \circ \gamma_j : t \longmapsto (0, \ldots, t, \ldots) \quad \text{i.e. } x^i(t) = \delta^i_j t$$

Kronecker delta

$$\Rightarrow \frac{d}{dt}(f \circ \gamma_j) = \frac{\partial \bar{f}}{\partial x^i} \frac{d}{dt}(\delta^i_j t)$$

$$= \frac{\partial}{\partial x^j} \bar{f}$$

We denote the natural vectors/coordinate vectors by

$$e_j = \frac{\partial}{\partial x^j} \quad \text{with } \underline{\text{components}} \ e_j{}^i = \delta^i_j$$

They form a basis for $T_p M$

__Example__ $\mathbb{R}^2 \ni P \xmapsto{\varphi} (x, y)$

coordinate vectors are $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$.

Polar coordinates: new chart $(\mathbb{R}^2, \psi)$

transition map

$$\varphi \circ \psi^{-1} : (x, y) \longmapsto \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \operatorname{Arctan} \dfrac{y}{x} \end{cases}$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$$

$$= \frac{x}{r}\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial \theta} \qquad \left[\frac{d \tan^{-1} x}{dx} = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} x & -y/r \\ y & x/r \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \Rightarrow \begin{cases} \dfrac{\partial}{\partial r} = \dfrac{1}{r}\left(x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\right) \\ \dfrac{\partial}{\partial \theta} = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x} \end{cases}$$

$r\frac{\partial}{\partial r}$ is infinitesimal dilation

infinitesimal rotation

* The space of vectors at P is represented by the vector space $\mathbb{R}^n \ni v^i$.

* Notice the above formula says

$$v_p(f) = \sum_i \underbrace{v^i}_{\text{vector}} \underbrace{\frac{\partial f}{\partial x^i}}_{\text{gradient}}$$

and vector $\cdot$ gradient is the $\mathbb{R}^n$ directional derivative.

Einstein summation convention

In nearly all our formulae, we can unambiguously discard the summation sign when repeated upper & lower indices appear so

$$v^i \frac{\partial}{\partial x^i} \equiv \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}$$

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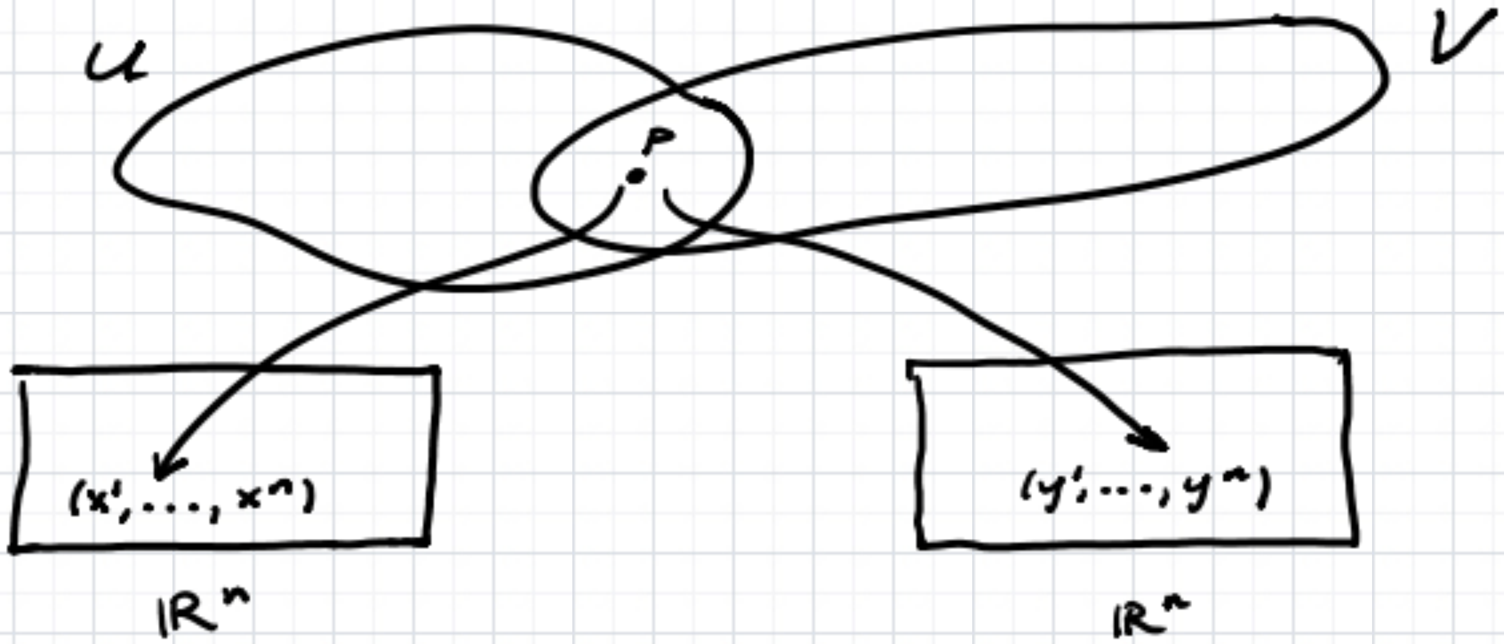

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