Lecture #4

Example

$$\begin{tikzcd} & (x,y) \arrow[dd, bend left] \\ \mathbb{R}^2 \arrow[ur, "\varphi"] \arrow[dr, "\psi"'] & \\ & (r,\theta) \end{tikzcd}$$

$$\psi\circ\varphi^{-1} : (x,y) \longmapsto \left(\sqrt{x^2+y^2}, \mathrm{Atan}\,\frac{y}{x}\right)$$

Relation b/w coordinate vectors (Jacobian)

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad \underline{\text{dilations}}$$

$$\frac{\partial}{\partial\theta} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \qquad \underline{\text{rotations}}$$

These operators are defined at all $p \in \mathbb{R}^2$,

so are vector fields not just vectors.

Algebra of vector fields on $\mathbb{R}^n$

$$\mathbb{R}^n \xrightarrow{\varphi} (x^1, \ldots, x^n)$$

Coordinate vectors $e_i = \frac{\partial}{\partial x^i}$

First moments $g^j{}_i = x^j \frac{\partial}{\partial x^i}$

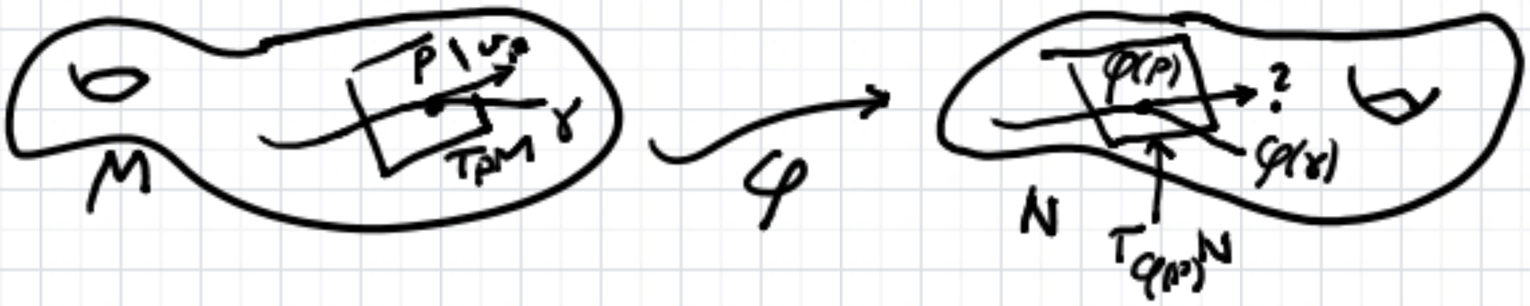
These differential operators close under commutation

$$\begin{cases} e_i e_j - e_j e_i \equiv [e_i, e_j] = 0 \\ g^j{}_i e_k - e_k g^j{}_i \equiv [g^j{}_i, e_k] = \delta^j_k e_k \\ g^j{}_i g^l{}_m - g^l{}_m g^j{}_i = [g^j{}_i, g^l{}_m] = \delta^l_i g^j{}_m - \delta^j_m g^l{}_i \end{cases}$$

This is an example of the Lie bracket and a Lie Algebra ($\mathbb{R}^n \ltimes \mathfrak{gl}(n)$). <u>More later</u>

<u>The Differential $d\varphi$</u>

<u>Idea</u> $\varphi : M \longrightarrow N$ differentiable



induces a map

$$d\varphi : T_pM \longrightarrow T_{\varphi(p)}N$$

The differential $d\varphi$ ($\equiv \varphi', \varphi_*, T\varphi, D\varphi, d\varphi_p$) is defined by

$$\underbrace{\left.\frac{d}{dt}\right|_0}_{\in\, T_pM} = \gamma'(0) \;\xmapsto{\;d\varphi\;}\; \underbrace{\left.\frac{d}{dt}\right|_0}_{\in\, T_{\varphi(p)}N} = (\varphi\circ\gamma)'(0)$$

Indeed

$$d\varphi(v_p)(\underbrace{f}_{\in\, \mathcal{D}_{\varphi(p)}(N)}) = v_p(\underbrace{f\circ\varphi}_{\in\, \mathcal{D}_pM})$$

$\frac{d}{dt}(f\circ\varphi\circ\gamma)$

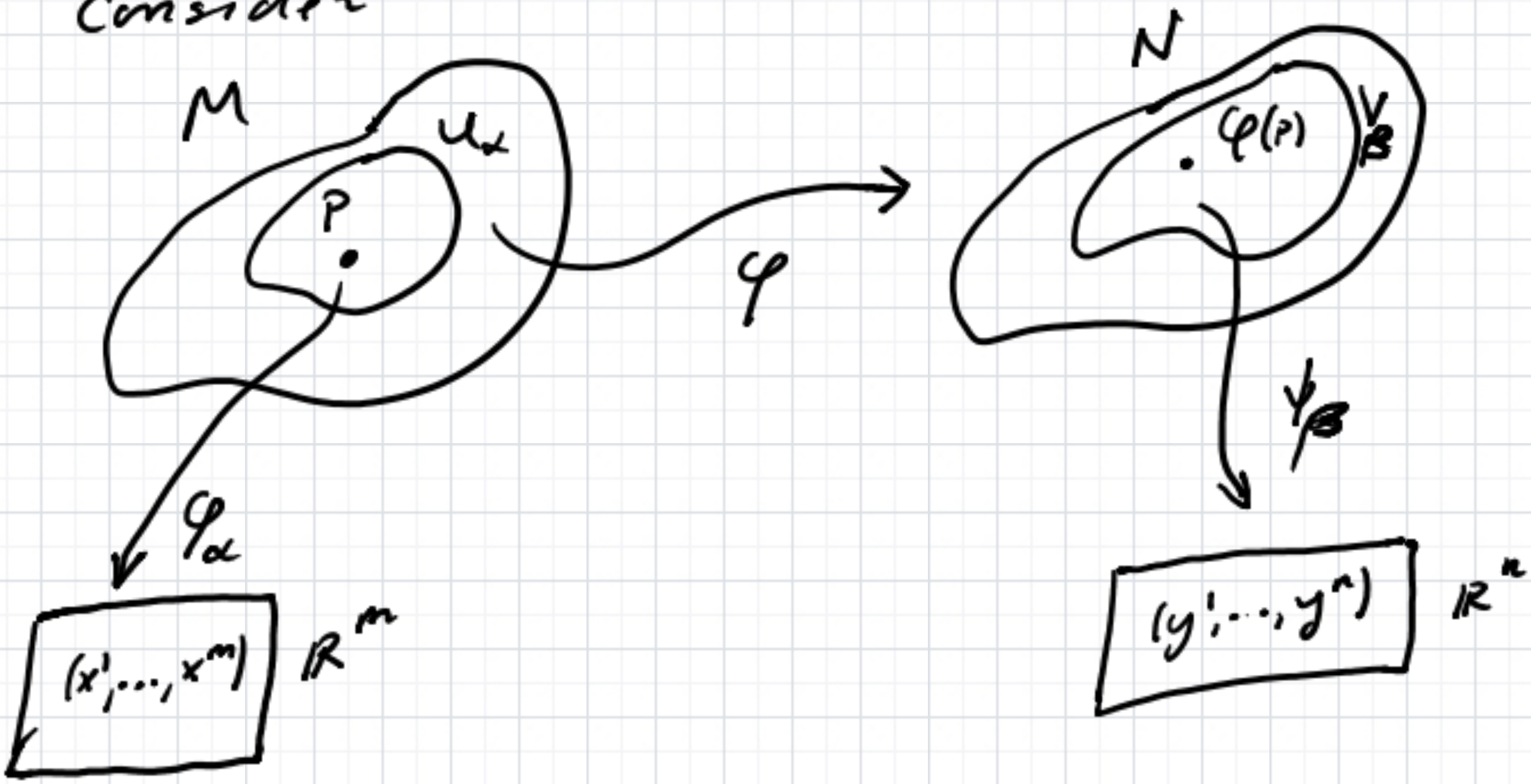
directional derivative of $f: N\to\mathbb{R}$ along $\varphi\circ\gamma$ in N

$\frac{d}{dt}(f\circ\varphi\circ\gamma)$

directional derivative of $f\circ\varphi: M\to\mathbb{R}$ along γ in M

The Jacobian of φ

Consider



i.e. $y(x) = \psi_\beta \circ \varphi \circ \varphi_\alpha(x)$

& T_pM has basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^m}\Big|_p \right\}$

$T_{\varphi(p)}N$ has basis $\left\{ \frac{\partial}{\partial y^1}\Big|_{\varphi(p)}, \dots, \frac{\partial}{\partial y^n}\Big|_{\varphi(p)} \right\}$

The Jacobian of φ is the $m \times n$ matrix representing $d\varphi$ in these bases.

I.e. $J = (J_i{}^a) \quad i = 1, \dots, m \; ; \; a = 1, \dots, n$ where

$$d\varphi\left(\frac{\partial}{\partial x^i}\Big|_p\right) = J_i{}^a \frac{\partial}{\partial y^a}\Big|_{\varphi(p)}$$

<u>Compute $J_i{}^a$</u>

$$\underbrace{d\varphi\left(\frac{\partial}{\partial x^i}\Big|_p\right)}_{\substack{\Cap \\ T_{\varphi(p)}N}} \underbrace{(f)}_{\substack{\Cap \\ \mathcal{S}_{\varphi(p)}M}} \equiv J_i{}^a \frac{\partial}{\partial y^a} \Big(\underbrace{f \circ \psi_\beta^{-1}}_{\tilde{f}(y)} \Big) \Big|_{\varphi(p)}$$

$$= \frac{\partial}{\partial x^i} \Big(\underbrace{f \circ \varphi \circ \varphi_\alpha^{-1}}_{f \circ \varphi(x)} \Big) \Big|_p = \frac{\partial}{\partial x^i} \left(f \circ \psi_\beta^{-1} \circ \psi_\beta \circ \varphi \circ \varphi_\alpha^{-1} \right)$$

Def^n of $d\varphi$

$$= \frac{\partial}{\partial x^i} \tilde{f}(y(x))$$

$$= \frac{\partial f \circ \psi_\beta^{-1}}{\partial y^a} \frac{\partial (\psi_\beta \circ \varphi \circ \varphi_\alpha^{-1})^a}{\partial x^i}$$

chain rule

<u>i.e.</u> $$\boxed{J_i{}^a = \frac{\partial}{\partial x^i} \left(\psi_\beta \circ \varphi \circ \varphi_\alpha^{-1} \right)^a \equiv \frac{\partial y^a}{\partial x^i}}$$

1-forms & the Cotangent space

Let $f : M \to \mathbb{R}$ be differentiable.

Then $df : T_pM \longrightarrow T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is a linear functional on the vector space T_pM.

The set of all <u>linear functionals</u> $\{T_pM \longrightarrow \mathbb{R}\} = T_p^*M$ the dual space to T_pM. It is called the cotangent space.

Coordinate differentials

Let $\varphi_\alpha :$ $\begin{array}{ccc} p & \longrightarrow & (x^1, \dots, x^n)(p) \\ \rotatebox[origin=c]{-90}{$\in$} & & \rotatebox[origin=c]{-90}{$\in$} \\ M & & \mathbb{R}^n \end{array}$ be a chart map at p.

We call $x^i(p)$ a coordinate function $x^i : U_\alpha \longrightarrow \mathbb{R}$

Regarding x^i as a differentiable map from manifolds $U_\alpha \longrightarrow \mathbb{R}$ we may consider dx^i

In fact, the coordinate differentials dx^i are <u>dual</u> to the coordinate vectors $\frac{\partial}{\partial x^i}$:

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial x^i}{\partial x^j} = \delta^i_j$$

and form a basis for T_p^*M.

Exercises ① Show that if $f : M \longrightarrow \mathbb{R}$

$$df\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial f}{\partial x^i}\Big|_p$$

i.e. the components of the gradient of f.

② Deduce the transformation properties of the components ω_i of a 1-form $\omega = \omega_i \, dx^i_p \in T^*_p M$ under changes of coordinates. Use this result to show $\sum_i \omega_i v^i$ (where v^i are the components of a vector at p) is invariant.

Observe $\bar{f} = \tilde{f} \circ \psi \circ \varphi^{-1}$

$$\Rightarrow \quad \frac{\partial \bar{f}}{\partial x^i} = \frac{\partial}{\partial x^i}\left[\tilde{f}(y(x))\right] \qquad \text{where } y(x) = \psi \circ \varphi^{-1}(x)$$

$$= \frac{\partial \tilde{f}}{\partial y^j}\frac{\partial y^j}{\partial x^i}$$

Chain rule

Thus if v_p has components v^i in chart $(U \cap V, \varphi)$ then

$$v_p(f) = v^i \frac{\partial \bar{f}}{\partial x^i}$$

$$= v^i \frac{\partial y^j}{\partial x^i}\frac{\partial \tilde{f}}{\partial x^j}$$

New components in the chart $(U \cap V, \psi)$ are

$$\boxed{\tilde{v}^j = \frac{\partial y^j}{\partial x^i} v^i} \qquad (*)$$

This rule is called the <u>transformation of a contravaraiant vector</u>. In many contexts the set of all components v^i with equivalence $(*)$ defined by $(*)$ is taken as the definition of a tangent vector