Lecture #5

The differential

$$\varphi : M \longrightarrow N$$

$$d\varphi : T_pM \to T_{\varphi(p)}N$$

via

$$d\varphi(v_p)(f) \quad = \quad v_p(f \circ \varphi)$$

$$\in T_pM \qquad \in D_{\varphi(p)}N \qquad\qquad \in D_pM$$

If $\quad f : M \longrightarrow \mathbb{R}$

$$df : T_pM \longrightarrow T_{\varphi(p)}\mathbb{R} \cong \mathbb{R} \qquad \text{linear functional}$$

$\Rightarrow \quad df \in T_p^*M \quad$ dual or <u>cotangent space</u>

Basis for T_p^*M

Given $(U_\alpha, \varphi_\alpha)$ then the i^{th} component of φ_α^{-1} maps $U_\alpha \longrightarrow \mathbb{R}$

$$x^i : p \longmapsto x^i(p) = \left(\varphi_\alpha^{-1}(p)\right)^i$$

denotes i^{th} slot

Viewing U_α as a manifold, consider

$$dx^i \in T_p^*M$$

Actually, $\{dx^i\}$ form a basis for T_p^*M, indeed the one <u>dual</u> to $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}$. They are called coordinate differentials/one-forms.

Compute

$$dx^i\left(\frac{\partial}{\partial x^j}\right)(f) = \frac{\partial}{\partial x^j}\left(f \circ x^i\right) = \frac{\partial x^i}{\partial x^j}\, f'$$

Making a choice of isomorphism $T_{x^i(p)}\mathbb{R} \cong \mathbb{R}$

yields $\boxed{dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j}$

<u>Notice</u> if $f : M \longrightarrow \mathbb{R}$

$$df\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial x^i}\Big|_p (f) = \frac{\partial \bar{f}}{\partial x^i}\Big|_p$$

$$\boxed{\begin{array}{l} M \longrightarrow \mathbb{R} \xrightarrow{g} \mathbb{R} \\ df\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}(g \circ f) \\ \qquad = g' \frac{\partial f}{\partial x} \end{array}}$$

Also $df(\cdot) = \omega_i\, dx^i(\cdot)$ [ω_i components of df in basis dx^i]

$\Rightarrow \omega_i = \frac{\partial \bar{f}}{\partial x^i}\Big|_p$ the <u>gradient</u> at p.

Directional derivative

Denote $\langle \cdot , \cdot \rangle : T_p^* M \times T_p M \longrightarrow \mathbb{R}$

via $\langle \omega, v \rangle \equiv \omega(v)$

Then $\langle dx^i, \frac{\partial}{\partial x^j} \rangle = \delta^i_j$

and if $f : M \to \mathbb{R}$ and $v_p = v^i \frac{\partial}{\partial x^i}$

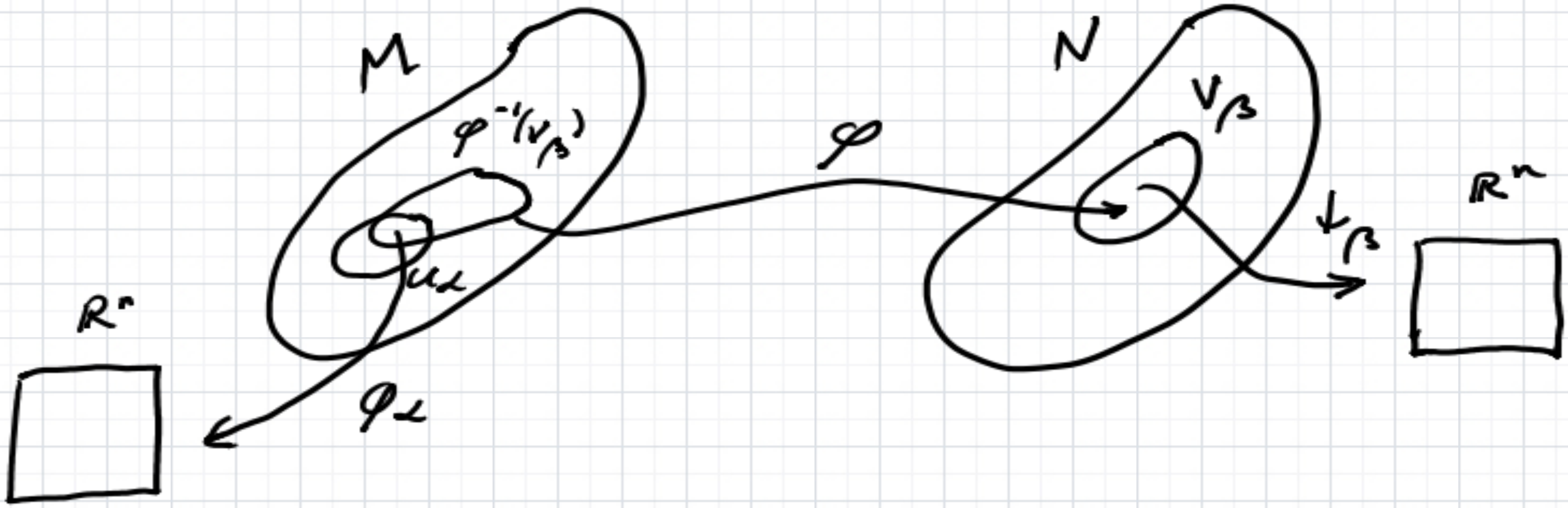
the canonical pairing

$$\langle df, v_p \rangle = v_p(f) = v^i \frac{\partial f}{\partial x^i}$$

the directional derivative

DIFFEOMORPHISMS

When are two differentiable manifolds the same?



If $\varphi : M \to N$ is <u>differentiable</u> and φ^{-1} <u>exists</u> and is also <u>differentiable</u> then

$$\left(U_\beta = \varphi^{-1}(V_\beta),\ \phi_\beta = \psi_\beta \circ \varphi\right)$$

is a <u>compatible</u> chart on M (just chase diagram using definition of differentiability).

We call $\varphi : M \to N$ a differentiable bijection $\bar{c}$ differentiable inverse a <u>diffeomorphism</u> and say M, N are diffeomorphic.

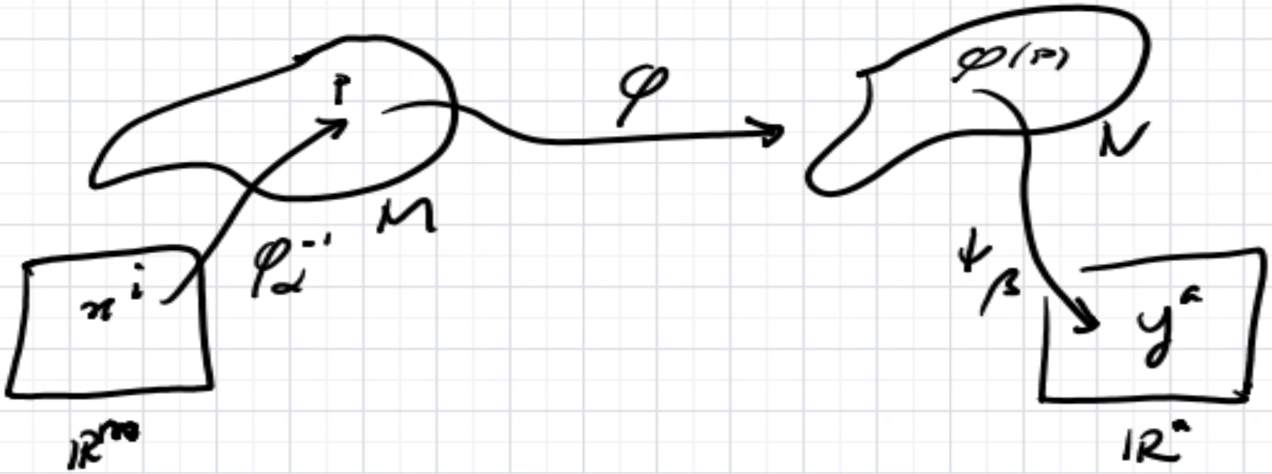
A <u>local diffeomorphism</u> at P is a differentiable map $\varphi : M \longrightarrow N$ s.t. φ is a diffeomorphism on open sets $U \ni P$, $V \ni \varphi(P)$ (viewed as manifolds).

<u>Recall</u> <u>Inverse function theorem</u>

$\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, differentiable with non-zero Jacobian determinant at P has a differentiable inverse in a neighbourhood of $\Phi(P)$.

Can apply inverse function theorem to

$$\psi_\beta \circ \varphi \circ \varphi_\alpha^{-1}, \quad \varphi \text{ differentiable}$$



Suppose $d\varphi : T_pM \to T_{\varphi(p)}N$ is an isomorphism,

then φ is a local isomorphism.

The converse is also true.

<u>Exercise</u>: Prove that $\varphi: M \to N$ a diffeomorphism

$\Rightarrow d\varphi: T_pM \longrightarrow T_{\varphi(p)}N$ is an isomorphism.

Observe $\bar{f} = \tilde{f} \circ \psi \circ \varphi^{-1}$

$$\Rightarrow \quad \frac{\partial \bar{f}}{\partial x^i} = \frac{\partial}{\partial x^i}\left[\tilde{f}(y(x)) \right] \qquad \text{where } y(x) = \psi \circ \varphi^{-1}(x)$$

$$= \frac{\partial \tilde{f}}{\partial y^j} \frac{\partial y^j}{\partial x^i}$$

Chain rule

Thus if v_p has components v^i in chart $(U \cap V, \varphi)$ then

$$v_p(f) = v^i \frac{\partial \bar{f}}{\partial x^i}$$

$$= v^i \frac{\partial y^j}{\partial x^i} \frac{\partial \tilde{f}}{\partial x^j}$$

New components in the chart $(U \cap V, \psi)$ are

$$\boxed{\tilde{v}^j = \frac{\partial y^j}{\partial x^i} v^i} \quad \text{——} \quad (*)$$

This rule is called the <u>transformation of a contravariant vector</u>. In many contexts the set of all components v^i with equivalence $(*)$ defined by $(*)$ is taken as the definition of a tangent vector