Lecture #6

Diffeomorphisms

Theorem If $\varphi : M \longrightarrow N$ is a local diffeomorphism at P, then

$$d\varphi : T_pM \longrightarrow T_{\varphi(p)}N$$

is a vector space isomorphism. Conversely, if $d\varphi : T_pM \longrightarrow T_{\varphi(p)}N$ is an isomorphism for some differentiable map at P, then φ is a <u>local</u> diffeomorphism at P.

For the converse apply the inverse function theorem to the differentiable map

$$\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad \text{via} \quad \Phi = \psi_\beta \circ \varphi \circ \varphi_\alpha^{-1}$$

where $(U_\alpha, \varphi_\alpha)$ & (V_β, ψ_β) are charts at P, $\varphi(P)$ and use the fact that the Jacobian is the matrix of the map $d\varphi$ in the coordinate basis.

For the forward direction, $d\varphi^{-1}$ is easily checked to be the required inverse ∎

The tangent bundle is a manifold

Let $TM = \{(P, v_p) : p \in M,\ v_p \in T_pM\}$

We call TM the tangent bundle. To see that TM is a manifold we need to build an atlas. Start with an atlas $\{(U_\alpha, \varphi_\alpha = (x^1_\alpha, \dots, x^n_\alpha))\}$ for M. At each $P \in M$, T_pM has a basis $\{\frac{\partial}{\partial x^i_\alpha}\big|_p\}$.

Take as chart map

$$\psi_\alpha : \underset{\substack{\cap \\ V_\alpha = \{(p, v_p) : p \in U_\alpha,\ v_p \in T_pM\}}}{\left(P,\ v_p = v^i \frac{\partial}{\partial x^i_\alpha}\Big|_p\right)} \longmapsto \underset{\substack{\cap \\ \mathbb{R}^{2n}}}{(x^1, \dots, x^n; v^1, \dots, v^n)}$$

From chart to chart $v^i_\alpha = \underbrace{\frac{\partial x^i_\beta}{\partial x^j_\alpha}}_{\substack{\text{The Jacobian of} \\ \text{a diffeomorphism } \mathbb{R}^n \to \mathbb{R}^n}} v^j_\beta$ is a linear transformation.

<u>Clearly differentiable</u>.

VECTOR FIELDS

"Sections of the tangent bundle".

A vector field is a map

$$\begin{aligned} X : M &\longrightarrow TM \\ \rotatebox{90}{$\in$}\ \ & \qquad\quad \rotatebox{90}{$\in$} \\ p &\longmapsto (p, v_p) \end{aligned}$$

and say X is differentiable if it is differentiable as a map b/w manifolds.

In a chart $\varphi^\alpha = (x^1, \dots, x^n)$

$$X(p) = X^i(p) \frac{\partial}{\partial x^i}\Big|_p$$

and differentiability of X requires differentiability of the component functions

$$p \longmapsto X^i(p)$$

Notice that a vector field defines a mapping

$$\underbrace{X : \mathcal{D}}_{\substack{\text{differentiable} \\ \text{functions on } M}} \longrightarrow \underbrace{\mathcal{F}}_{\substack{\text{functions} \\ \text{on } M}} \quad \text{via } (Xf)(p) = X^i(p) \frac{\partial f}{\partial x^i}(p)$$

$$= (X(f))(p)$$

the directional derivative of f in the direction of X at p.

Suppose X, Y are differentiable vector fields and $f : M \to \mathbb{R}$ is C^∞ then $X(f)$ and $Y(f)$ are both differentiable functions. Hence we may consider $X(Y(f))$.

Calculate in a chart

$$X(Y(f)) = X\left(Y^i \frac{\partial \bar{f}}{\partial x^i}\right)$$

$$= X^j \frac{\partial}{\partial x^j}\left(Y^i \frac{\partial \bar{f}}{\partial x^i}\right)$$

(should put a bar over the Y^i)

$$= X^j Y^i \frac{\partial^2 \bar{f}}{\partial x^j \partial x^i} + X^j \frac{\partial Y^i}{\partial x^j} \frac{\partial \bar{f}}{\partial x^i}$$

not a vector field

Acts like a vector field

$$\Rightarrow (XY - YX)(f) \equiv [X, Y](f)$$

$$= \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\right) \frac{\partial \bar{f}}{\partial x^i}$$

is a vector field. The vector field $[X, Y]$ is called the (Lie) bracket of X and Y.
It is differentiable when X & Y are.

Properties of $[\cdot, \cdot] : \Lambda^2 \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$

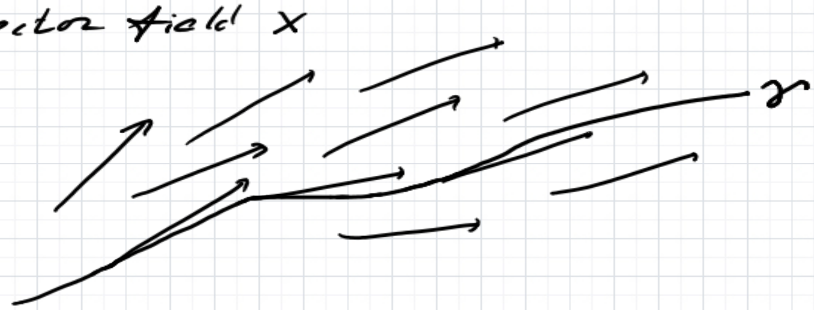
(i) $[X, Y] = -[Y, X]$ antisymmetry

(ii) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ Linear

(iii) $[[X, Y], Z] + \text{cyclic} = 0$ Jacobi

(iv) $[fX, gY] = fg[X, Y] + f X(g) Y - g Y(f) X$ Leibnitz

Given a vector field X



we can search for integral curves such that for any $p \in \gamma$, $v_p = \gamma'(t_p)$ where $\gamma(t_p) = p$

To solve this problem we must study a system of ODE: In coordinates x^i

$$X = X^i \frac{\partial}{\partial x^i} \quad \text{and} \quad x^i(t) = (\varphi_\alpha \circ \gamma)^i(t)$$

so components are

$$X^i(\gamma(t)) = \frac{dx^i(t)}{dt}$$

with some initial condition, the start of the integral curve $x^i(0) = \varphi^i_\alpha(p)$.

Small time existence of solutions to the system $X^i(x) = \dot{x}^i$ is guaranteed by Picard's theorem.

The curves $\gamma_t(p)$ allow us to define a Lie derivative $(\mathcal{L}_X f)(p) = \lim_{t \to 0} \frac{f(\gamma_t(p)) - f(p)}{t}$

Exercises ① Find out what it means for a manifold to be orientable.

Give examples of an orientable and a non-orientable differentiable structure.

② Check the properties of the Lie bracket.