Lecture #7 Lie dragging & the Lie Derivative

Vector fields define integral curves

compare values of objects as you move along flows.

→ Lie derivative $\mathcal{L}_X$ w.r.t. a vector field

→ Not parallel transport/covariant derivative (needs metric).

Definition A curve γ whose tangent vector

$$\frac{d}{dt}\Big|_p = X(p) \qquad \forall\, p \in \mathrm{Im}(\gamma)$$

for some vector field X is called an integral curve of X. Sometimes called the trajectory of a flow.

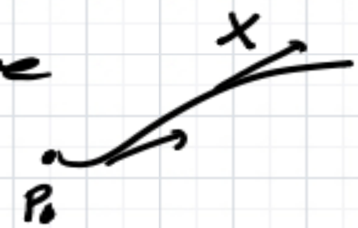
To find integral curves for

$$X = X^i \frac{\partial}{\partial x^i}$$

in some chart, we need to solve a system of ODEs

$$X^i(\gamma(t)) = \frac{dx^i}{dt} \quad \text{where } x^i = (\varphi_\alpha \circ x)^i(t)$$

subject to an initial condition $x^i(0) = \varphi_\alpha^i(p_0)$ where p_0 is the start of the curve



Certainly can't generically expect global solutions (e.g 

this flow), but Picard's theorem guarantees at least short time existence.

Theorem If X is a C^r vector field, then $\forall\, p \in M$ $\exists_1$ an integral curve of X

$$\gamma : t \longrightarrow \gamma(t, p)$$

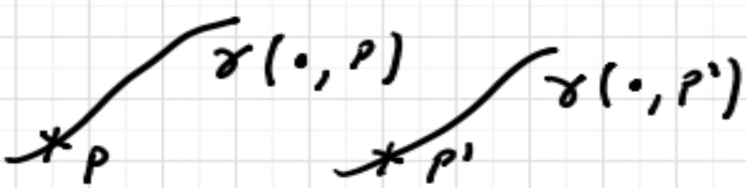
such that

(i) $\gamma(0, p) = p \qquad \forall\, p \in M$

(ii) $\gamma(t, p)$ is defined and differentiable on some interval $0, t \in I(p) \subset \mathbb{R}$

(iii) $\gamma(t, \gamma(t', p)) = \gamma(t + t', p) \qquad t, t', t+t' \in I(p)$

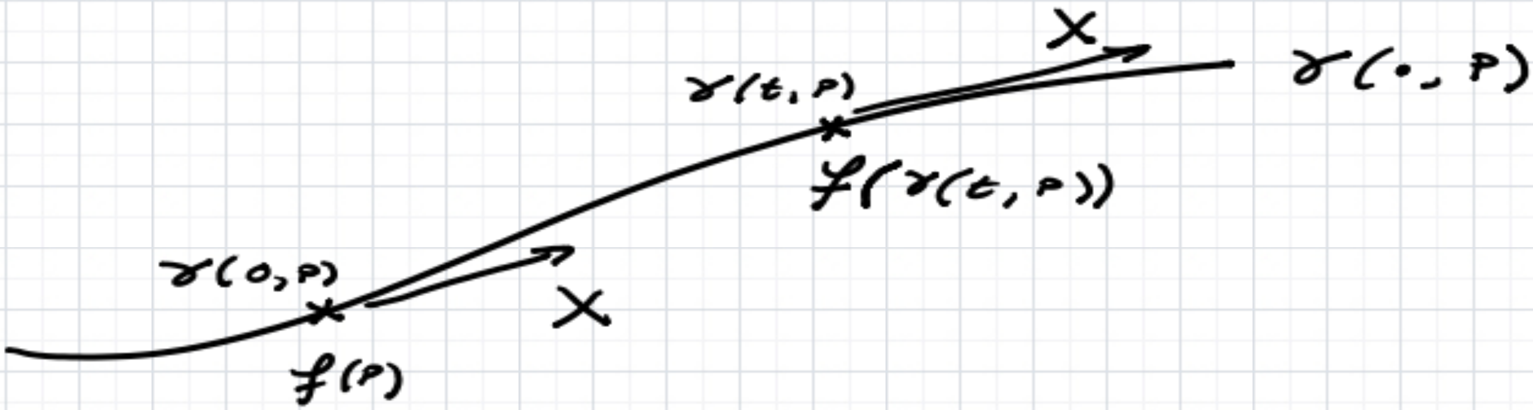
Notice this is a family of curves



(iii) is case where p' lies on $\gamma(\cdot, p)$.

Proof: Choquet-Bruhat, Analysis Manifolds & Physics. II.D DEs

The Lie Derivative on Functions



<u>Define</u> $$(\mathcal{L}_X f)(p) = \lim_{t\to 0} \frac{f(\gamma(t,p)) - f(p)}{t}$$

$$= \lim_{t\to 0} \frac{f(\gamma(t,p)) - f(\gamma(0,p))}{t}$$

$$= \frac{d}{dt} \frac{f(\gamma(t,p))}{dt}$$

$$= \frac{d}{dt}\left(f\circ\varphi^{-1}\ \varphi\circ\gamma\right)$$

$$= \dot{x}^i(0)\, \frac{\partial \bar{f}}{\partial x^i}$$

$$= v_p(f)$$

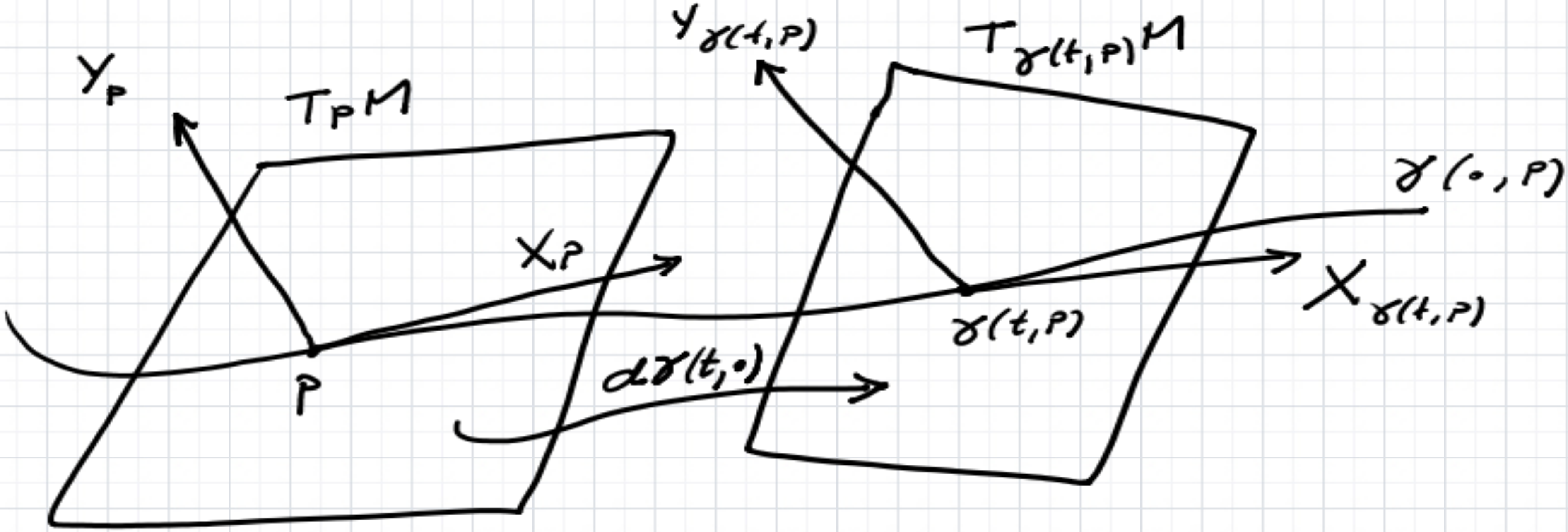
<u>i.e.</u> $$\boxed{\mathcal{L}_X(f) = X(f)}$$

Recall The bracket obeys

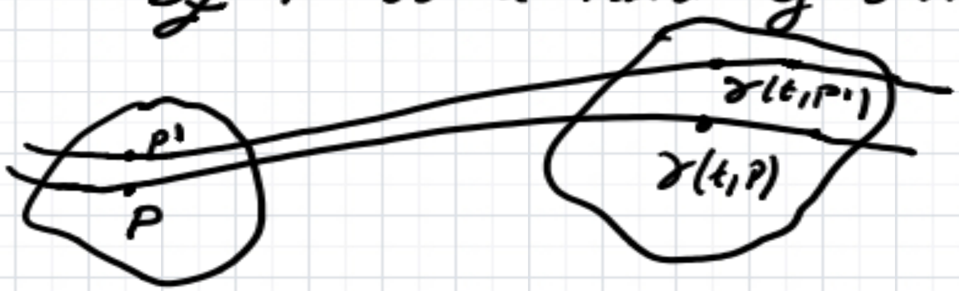
$$[X, fY] = X(f)Y + f[X,Y]$$
$$= \mathcal{L}_X(f)Y + f[X,Y]$$

Hence we suspect $\mathcal{L}_X = [X, \cdot]$

The Lie Derivative on Vectors



Idea Use $\gamma(t,\cdot)$ as a map b/w a neighborhood of p to a nhd. of $\gamma(t,p)$, its inverse is $\gamma(-t,\cdot)$.



Then Lie derivative of a vector field Y along X

$$\boxed{(\mathcal{L}_X Y)(p) = \lim_{t\to 0} \frac{(d\gamma(-t,\cdot))(Y(\gamma(t,p))) - Y(p)}{t}}$$

Notice the $-t$, we are "dragging" the vector $Y(\gamma(t,p))$ back to T_pM so we can compare it to $Y(p)$

Claim $\mathcal{L}_X Y = [X,Y]$. Calculate:

$$(\mathcal{L}_X Y)(f)(p) = \lim_{t\to 0} \frac{Y(\gamma(t,p))(f) - (d\gamma(t,\cdot))(Y(p))(f)}{t}$$

$\times\, d\gamma(t,\cdot)$ b/c $\lim_{t\to 0} d\gamma(t,\cdot) = d\gamma(0,\cdot) = \mathrm{Id}$

acts with dragged vector

acts on dragged f^n

$$= \lim_{t\to 0} \frac{Y(\gamma(t,p))(f) - Y(p)(f\circ\gamma(t,p))}{t}$$

def$^{\underline{n}}$ of differential

<u>Lemma</u> $f \circ \gamma(t,\cdot) - f = t\, g(t,\cdot)$ for some $g(t,\cdot): M \to \mathbb{R}$ differentiable and

$$g(0,\cdot) = X(f)$$

The boundary condition follows by taking $t \to 0$

To construct $g(t,\cdot)$ write

$$\begin{aligned} f\circ\gamma(t,\cdot) - f &= f\circ\gamma(t,\cdot) - f(0,\cdot) \\ &= \int_0^t ds \frac{d}{ds} f\circ\gamma(s,\cdot) \\ &= t \underbrace{\int_0^1 ds \frac{d}{d(st)} \gamma(st,\cdot)}_{g(t,\cdot)} \end{aligned}$$

Hence

use g & linearity of Y

$$(\mathcal{L}_X Y)(f)(p) = \lim_{t\to 0} \frac{Y(\gamma(t,p))(f)(p) - [Y(p)(f)(p) - Y(p)(t\, g(t,\cdot))]}{t}$$

$$= \lim_{t\to 0} \frac{Y(\gamma(t,p))(f)(p) - Y(p)(f)(p)}{t} - \lim_{t\to 0} Y(p)(g(t,\cdot))(p)$$

$$= X(p)(Y(f)) - Y(p)(X(f))$$

Exercises ① Does $L_X(Y(f)) = (L_X Y)(f)$?

② What can you say about the commutator of coordinate vector fields?