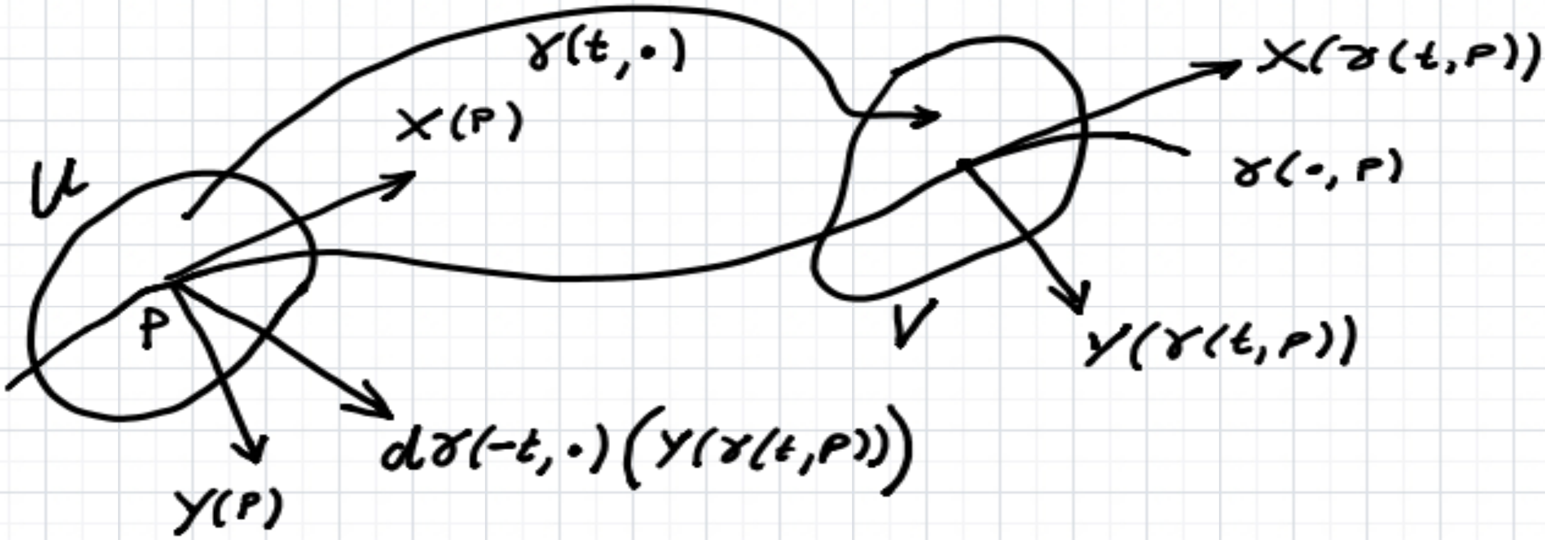
Lecture #8 The Lie Derivative on Vectors



$$\boxed{\begin{aligned} \mathcal{L}_X Y\Big|_p &= \lim_{t\to 0} \frac{d\gamma(-t,\cdot)(Y(\gamma(t,p))) - Y(p)}{t} \\ &= \lim_{t\to 0} \frac{Y(\gamma(t,p)) - d\gamma(t,\cdot)(Y(p))}{t} \end{aligned}}$$

Acting on $f: V \to \mathbb{R}$ using def^{n} of differential

$$\big(\mathcal{L}_X Y(f)\big)(p) = \lim_{t\to 0} \frac{Y(\gamma(t,p))(f) - Y(p)(f\circ\gamma(t,\cdot))}{t}$$

Define $g(t,\cdot)$ by

$$t\, g(t,\cdot) = f \circ \gamma(t,\cdot) - f$$

and notice $g(0,\cdot) = \lim_{t \to 0} \dfrac{f \circ \gamma(t,\cdot) - f}{t} = X(f)$

Thus

$$\big(\mathcal{L}_X Y(f)\big)(p) = \lim_{t\to 0} \frac{Y(\gamma(t,p))(f) - Y(p)(f) - Y(p)(t\,g(t,\cdot))}{t}$$

Y(p) is Linear

$$= \lim_{t\to 0} \frac{[Y(f)](\gamma(t,p)) - [Y(f)](p)}{t} - \lim_{t\to 0} Y(p)(g(t,\cdot))$$

This is $X(p)$
acting on the fn $Y(f)$

$$= X(Yf)\big|_p - Y(g(0,\cdot))$$

$$= [XY - YX](f)\big|_p$$

i.e. $\mathcal{L}_X Y = [X, Y]$ //

Examples

Coordinate Vector Fields

In a given chart $(U_\alpha, \varphi_\alpha = (x^1, \dots, x^n))$, we define

$$\begin{aligned} e_i : U_\alpha &\longrightarrow TM \\ \rotatebox{90}{$\in$}\ \ & \qquad\ \ \rotatebox{90}{$\in$} \\ p &\longmapsto \frac{\partial}{\partial x^i}\Big|_p \end{aligned}$$

Often call $e_i = \frac{\partial}{\partial x^i}$

Then $\mathcal{L}_{e_i} e_j = [e_i, e_j] = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$

Conformal Boosts in $\mathbb{R}^2$

$$\begin{cases} K_x = -2x\,(x\partial_x + y\partial_y) + (x^2+y^2)\,\partial_x \\ K_y = -2y\,(x\partial_x + y\partial_y) + (x^2+y^2)\,\partial_y \end{cases}$$

$$[K_x, K_y] = 0 = [e_x, e_y]$$

$$[K_x, e_x] = 2\,(x\partial_x + y\partial_y) = 2D \qquad \text{dilation}$$

$$[K_x, e_y] = 2\,(x\partial_y - y\partial_x) = 2\frac{\partial}{\partial\theta} \qquad \text{rotation}$$

$\{e_x, e_y, D, \frac{\partial}{\partial\theta}, K_x, K_y\}$ generate $so(3,1) = co(\mathbb{R}^2)$

Lie Derivative of 1-forms

The cotangent bundle T^*M inherits a differentiable structure from M in the same way TM does.

A **differentiable 1-form** α is a differentiable map

$$\begin{array}{ccc} M & \xrightarrow{\quad\alpha\quad} & T^*M = \{(p,\theta) : p \in M,\ \theta \in T_p^*M\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ p & \longmapsto & (p, \alpha_p) \end{array}$$

To compare 1-forms at different point in M we need to find a map b/w cotangent spaces induced by

Such a map exists and is called the **pullback**:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr] & N \arrow[d, "f"] \\ & P \end{tikzcd}$$

$\varphi^*(f) = f \circ \varphi$

Application to 1-forms

Let $\varphi : M \longrightarrow N$, and ω be a 1-form on N, then:

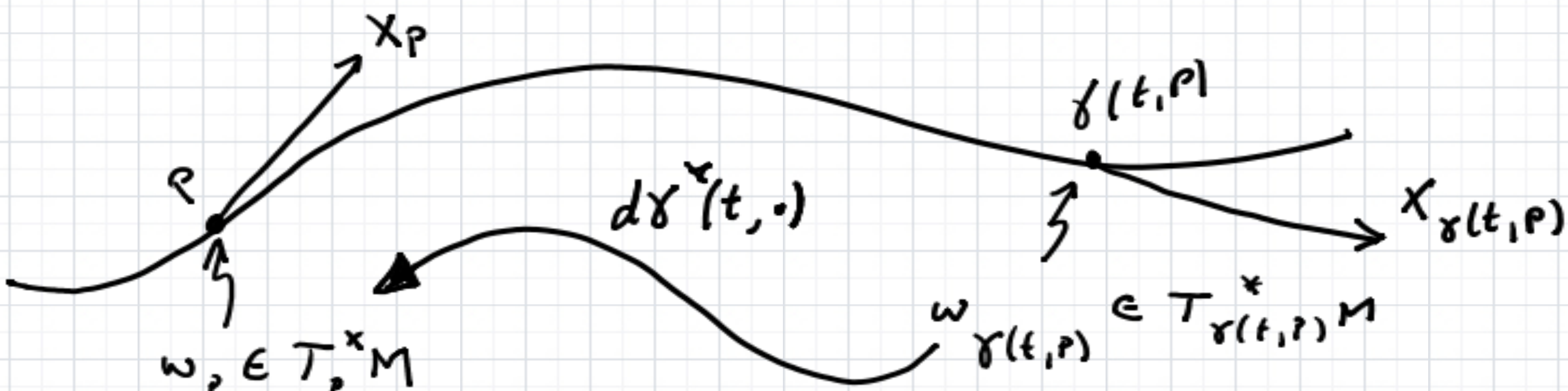
$$\begin{tikzcd} T_p M \arrow[r, "d\varphi"] \arrow[dr, "d\varphi^*(\omega) = \omega\circ d\varphi"'] & T_{\varphi(p)} N \arrow[d, "\omega_{\varphi(p)}"] \\ & \mathbb{R} \end{tikzcd}$$

is <u>linear functional</u>

Most often call $d\varphi^*$ simply φ^*

$$\varphi^* : T^*_{\varphi(p)} N \longrightarrow T^*_p M$$
$$\omega \longmapsto \omega\circ d\varphi$$

The Lie derivative



Then Lie derivative of a one form ω along X

$$(\mathcal{L}_X \omega)(p) = \lim_{t \to 0} \frac{(d\gamma^*(t,\cdot))(\omega(\gamma(t,p))) - \omega(p)}{t}$$

In components, if $X = X^i \partial_i$ in some coordinate chart and $\omega = \omega_i dx^i$

$$\mathcal{L}_X \omega = (X^j \partial_j \omega_i + (\partial_i X^j)\omega_j)\, dx^i$$

Directional derivative term "orbital"

"Rotates" cotangent space "spin"

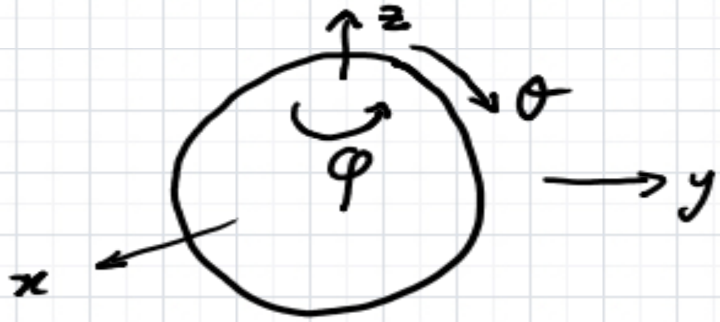
Mnemonic: use the bracket notation

$$[X^i \partial_i, \omega_j dx^j] = [X^i \partial_i, \omega_j] dx^j + \omega_j [X^i \partial_i, dx^j]$$

$$= X^i (\partial_i \omega_j) dx^j + \omega_j\, d(X^i \underbrace{\partial_i x^j}_{\delta_i^j})$$

$$= X^i (\partial_i \omega_j) dx^j + \omega_j\, dX^j$$ agrees w/ above

Example S^2 with coordinates (θ, φ)



Diffeomorphisms corresponding to rotations generated by rotations about x, y, z axes are

$$L_x = -\sin\varphi\, \partial_\theta - \cot\theta \cos\varphi\, \partial_\varphi$$

$$L_y = +\cos\varphi\, \partial_\theta - \cot\theta \sin\varphi\, \partial_\varphi$$

$$L_z = \partial_\varphi$$

"Zweibein" $e^1 = d\theta \qquad e^2 = \sin\theta\, d\varphi$.

Claim $ds^2 = (e^1)^2 + (e^2)^2$ is "invariant"

Ex L_z is obvious b/c $\mathcal{L}_{L_z} e^1 = \mathcal{L}_{L_z} e^2 = 0$

ds^2 is a $\binom{0}{2}$-tensor called the metric tensor.

It defines the canonical metric on the sphere inherited from $\mathbb{R}^3$.

Exercises ① Let $X = X^i \partial_i$ & $\omega = \omega_i dx^i$,

show

$$\mathcal{L}_X \omega = \left(X^i \partial_i \omega_j + (\partial_j X^i) \omega_i \right) dx^j$$

② Check invariance of ds^2 as defined for the sphere above. Also compute the algebra of L_x, L_y, L_z w.r.t. the bracket $[\,,\,]$. Explain the result geometrically.