Lecture #8 THE PULLBACK

$$\gamma(t,\cdot) : U \ni p \longrightarrow V = \gamma(t,U)$$

$$\begin{tikzcd}
d\gamma(t,\cdot) : T_pM \arrow[r] \arrow[dr] & T_{\gamma(t,p)}M \arrow[d, "\omega_{\gamma(t,p)} \in T^*_{\gamma(t,p)}M"] \\
& \mathbb{R}
\end{tikzcd}$$

$$T_p^*M \ni d\gamma(t,\cdot)^* = \omega_{\gamma(t,p)} \circ d\gamma(t,\cdot)$$

often denoted $\gamma^*(t,\cdot)$

Lie derivative on forms

$$(\mathcal{L}_X \omega)(p) = \lim_{t \to 0} \frac{\left(d\gamma^*(t,\cdot)\right)\left(\omega(\gamma(t,p))\right) - \omega(p)}{t}$$

In components, if $X = X^i \partial_i$ in some coordinate chart and $\omega = \omega_i dx^i$

$$\mathcal{L}_X \omega = (X^j \partial_j \omega_i + (\partial_i X^j)\omega_j)\, dx^i$$

Directional derivative term "orbital"

"Rotates" cotangent space "spin"

<u>Mnemonic</u>: use the bracket notation

$$[X^i \partial_i , \omega_j dx^j] = [X^i \partial_i , \omega_j] dx^j + \omega_j [X^i \partial_i , dx^j]$$
$$= X^i (\partial_i \omega_j) dx^j + \omega_j d(X^i \underbrace{\partial_i x^j}_{\delta_i^j})$$
$$= X^i (\partial_i \omega_j) dx^j + \omega_j dX^j \quad \underline{\text{agrees } \bar{c} \text{ above}}$$

<u>Example</u> $K_x = -2x(x\partial_x + y\partial_y) + (x^2+y^2)\partial_x$

$$\begin{cases} \mathcal{L}_{K_x} dx = d\left(-2x^2 \frac{\partial x}{\partial x} + (x^2+y^2)\frac{\partial x}{\partial x}\right) = d(-x^2+y^2) = -2(x dx - y dy) \\ \mathcal{L}_{K_x} dy = d\left(-2xy \frac{\partial y}{\partial y}\right) = -2(y dx + x dy) \end{cases}$$

$\underset{\text{Leibnitz}}{\Longrightarrow}$ $$\mathcal{L}_{K_x}(dx^2 + dy^2) = -4dx(xdx - ydy) - 4dy(ydx + xdy)$$
$$= -4x(dx^2 + dy^2)$$

$ds^2 = dx^2 + dy^2$ is the <u>metric</u> on $\mathbb{R}^2$ and $\mathcal{L}_Y ds^2 = \Omega d^2s$

is called a <u>conformal isometry</u>

Tensor Calculus

Tensors at a point — defined exactly as in $\mathbb{R}^n$:

<u>1-form</u> $\omega_p : T_pM \longrightarrow \mathbb{R}$

$$v_p \longmapsto \omega_p(v_p) = \langle \omega_p, v_p \rangle$$

<u>vector</u> $v_p : T_p^*M \longrightarrow \mathbb{R}$

$$\omega_p \longmapsto \omega_p(v_p) = \langle \omega_p, v_p \rangle$$

<u>i.e.</u> view v_p as an element of $(T_p^*M)^* = T_pM$

A tensor X of type $\binom{n}{m}$ is a <u>multi-linear</u> function

$$\underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{n-\text{ times}} \otimes \underbrace{T_pM \otimes \cdots \otimes T_pM}_{m \text{ times}} \xrightarrow{X} \mathbb{R}$$

$$\equiv \otimes^n T_p^*M \qquad\qquad \equiv \otimes^m T_pM$$

i.e. X eats $n-1$ forms & m-vectors and <u>spits</u> out a number.

<u>Ex</u> $v_p \in T_pM$ is a tensor of type $\binom{1}{0}$

$\omega_p \in T_p^*M$ —— " —— $\binom{0}{1}$

Notice:

$$X : \otimes^n T_p^* M \otimes \otimes^m T_p M \longrightarrow \mathbb{R}$$

but

$$X \in \otimes^n T_p M \otimes \otimes^m T_p^* M$$

Explicitly multilinearity means

$$X(\underbrace{\cdots, \lambda v + \mu u, \cdots}_{n\text{-slots for vectors}} ; \underbrace{\cdots\cdots}_{m\text{-slots for 1-forms}}) = \lambda X(\cdots, v, \cdots ; \cdots) + \mu X(\cdots u \cdots ; \cdots)$$

and similarly in the 1-form slots.

Remark: The space of $\binom{n}{m}$-tensors at p is a vector space of dimension $(\dim M)^{n+m}$

Example $\binom{0}{2}$ tensors. These act on pairs of vectors. Let $\{e_i\}$ be any basis (e.g. $\{\frac{\partial}{\partial x^i}|_p\}$) for T_pM and $\{\theta^i\}$ be the dual basis (e.g. $\{dx^i|_p\}$) so that

$$\theta^i(e_j) = \delta^i_j$$

Let $v_p = v^i e_i$, $u_p = u^i e_i \in T_pM$ and $\omega \in \otimes^2 T_p^* M$ be a $\binom{0}{2}$-tensor. Then by multi-linearity

$$\omega(v, u) = v^i u^j \, \omega(e_i, e_j)$$

Now $\theta^i(v_p) = v^j \theta^i(e_j) = v^j \delta^i_j = v^i$

so $(\underbrace{\theta^i \otimes \theta^j}_{\in \otimes^2 T_p^* M})(v,u) = v^i u^j$

Hence $\omega(v,w) = (\underbrace{\theta^i \otimes \theta^j}_{n^2 \text{ basis vectors for } \otimes^2 T_p^* M})(v,u) \cdot \underbrace{\omega(e_i, e_j)}_{\text{components of } \omega \text{ in this basis}}$

Calling $\omega(e_i, e_j) = \omega_{ij}$ we have

$$\boxed{\omega(v,w) = v^i u^j \omega_{ij}}$$

RHS may be familiar!

In general for a $\binom{n}{m}$ tensor ω acting on n 1 forms $\alpha^1, \ldots, \alpha^n$ & m vectors $v_1, \ldots, v_m$

$$\omega(\alpha^1, \ldots, \alpha^n; v_1, \ldots, v_m) = \alpha^1_{i_1} \cdots \alpha^n_{i_n} v_1^{j_1} \cdots v_m^{j_m} \omega^{i_1 \cdots i_n}_{j_1 \cdots j_m}$$

where $\boxed{\omega^{i_1 \cdots i_n}_{j_1 \cdots j_m} = \omega(\theta^{i_1}, \ldots, \theta^{i_n}; e_{j_1}, \ldots, e_{j_m})}$

Change of basis & $Gl(\mathbb{R}, n)$

Recall the operator

$$T_pM \ni v_p : \mathcal{D}(P) \longrightarrow \mathbb{R}$$

is unchanged by changing coordinate charts, but its components undergo

$$v^i \longmapsto \tilde{v}^i = \frac{\partial y^i}{\partial x^j} v^j$$

$$g \in Gl(n, \mathbb{R}) \qquad v \in \mathbb{R}^n$$

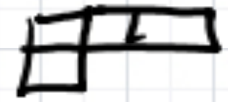
I.e. $v \mapsto gv$ so v is the fundamental representation of $Gl(n, \mathbb{R})$. Sometimes called $\underline{n}$.

Similarly the components of a 1-form ω_p in the dual basis $\omega_p = \omega_i dx^i$, $\omega_i \in (\mathbb{R}^n)^*$ transform as

$$\omega \longmapsto \bar{\omega} = (g^{-1})^t \omega$$

This is the conjugate representation (or $\bar{n}$).

Notice $\omega_p(v_P) = \omega_i v^i = \omega^t v \longmapsto (g^{-t}\omega)^t g v = \omega^t v$ is invariant

This means that tensors can be classified by decomposing them as irreducible $Gl(n,\mathbb{R})$ representations. (These are labeled by Young diagrams  etc... corresponding the action of the Weyl group S^n of $Gl(n)$.)

<u>Multiplication of tensors</u> If X and Y are $\binom{n}{m}$ and $\binom{p}{q}$ tensors at P, we can define a $\binom{n+p}{m+q}$ tensor $X \otimes Y$ by

$$(X \otimes Y)(\cdot,\cdot) = X(\cdot)\, Y(\cdot)$$

concatenating entries

The components of $X \otimes Y$ are just the <u>product</u> of those of X & Y.

Exercises: ① Find out anything you can about irreducible representations of $GL(n)$ and write a short essay on your findings.

Discuss the irreducible parts of some simple $\binom{n}{m}$ tensors (i.e. small n & m) as examples.

② How would you define a differentiable tensor field and the tensor bundle $T^{\sim}M$ over M?