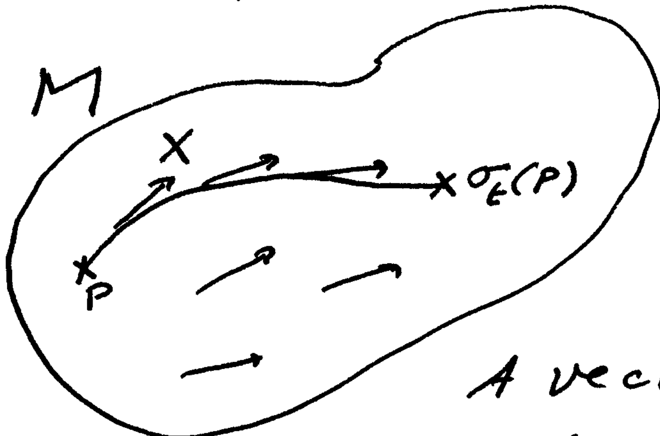
Lie Derivatives (See, e.g., Analysis Manifolds and Physics, Choquet-Bruhat, De-Witt-Morette, Dillard-Bleick pp 147-150)

① The fluffy idea:



A vector field defines a flow $\sigma_t : M \to M$

Want to use this flow to move stuff (vectors, tensors, functions, forms) around the manifold M and see what happens at small "times" t.

② Example functions

Let $f : M \longrightarrow \mathbb{R}$

and $X \in \mathfrak{X}(M)$ (a vector field).

Then
$$\mathcal{L}_X f \Big|_P \equiv \lim_{t \to 0} \frac{f(\sigma_t(P)) - f(P)}{t}$$

The Lie derivative operator

An example of the general definition of $\mathcal{L}_X$

$$= \frac{d}{dt} f(\sigma_t(P))\Big|_{t=0}$$

$$= X(f)\Big|_P$$

But $\sigma_t(P)$ is a path through P tangent to X

i.e. $\boxed{\mathcal{L}_X(f) = X(f)}$ The Lie derivative of a function f w.r.t. a vector field X is just the directional derivative.

③ WARNING — THE COVARIANT DERIVATIVE IS NOT THE LIE DERIVATIVE

④ Concrete, computationally useful, uninformative coordinate expressions:

Suppose

$$X = X^\mu \frac{\partial}{\partial x^\mu}$$

and

$$Y^{\mu_1 \ldots \mu_s}{}_{\nu_1 \ldots \nu_t}$$

are components of a type $\binom{s}{t}$ tensor,

then

scalar / "orbital" term

$$\mathcal{L}_X Y^{\mu_1 \ldots \mu_s}{}_{\nu_1 \ldots \nu_t} = X^\mu \frac{\partial Y^{\mu_1 \ldots \mu_s}{}_{\nu_1 \ldots \nu_t}}{\partial x^\mu}$$

$$+ \sum_{i=1}^{t} \left(\frac{\partial X^\mu}{\partial x^{\nu_i}} \right) Y^{\mu_1 \ldots \mu_s}{}_{\nu_1 \ldots \mu \ldots \nu_t}$$

slot i

slot j

spin terms

$$- \sum_{j=1}^{s} \frac{\partial X^{\mu_j}}{\partial x^\mu} Y^{\mu_1 \ldots \mu \ldots \mu_s}{}_{\nu_1 \ldots \nu_t}$$

(5) Mnemonic:

To act on

$$Y^{\mu_1 \cdots \mu_s}_{\nu_1 \cdots \nu_t} \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_s}} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_s}$$

use

$$\mathcal{L}_X = \left[X^\mu \frac{\partial}{\partial x^\mu} , \cdot \right]$$

where the operator

$[🍎, \cdot]$ obeys Leibnitz's rule

an apple!

and

* $\left[X^\mu \frac{\partial}{\partial x^\mu} , f(x) \right] \equiv X(f) = X^\mu \frac{\partial f}{\partial x^\mu}$

any function of x^μ, e.g. $Y^{\mu_1 \cdots \mu_s}_{\nu_1 \cdots \nu_t}(x)$

* $\left[X^\mu \frac{\partial}{\partial x^\mu} , \frac{\partial}{\partial x^\nu} \right] \equiv - \frac{\partial X^\mu}{\partial x^\nu} \frac{\partial}{\partial x^\mu}$

* $\left[X^\mu \frac{\partial}{\partial x^\mu} , dx^\nu \right] \equiv d\left(X^\mu \frac{\partial x^\nu}{\partial x^\mu} \right) = d\left(X^\mu \delta^\nu_\mu \right)$

$$= d(X^\nu) = \frac{\partial X^\nu}{\partial x^\mu} dx^\mu$$

⑥ The big Formaggio

Given a flow $\sigma_t : M \longrightarrow M$ on functions

$$\sigma_t^*(f) \equiv f \circ \sigma_t$$

For vectors we can define

$$\sigma_{t*}(v(f)) \equiv v(\sigma^* f) = v(f \circ \sigma_t)$$

a function

For one forms

$$\sigma_t^* \omega(v) \equiv \omega(\sigma_{t*}(v))$$

a vector

The operator $\mathcal{L}_X$ is then

$$\mathcal{L}_X = \frac{d\sigma_t^{\#}}{dt}$$

where the # stands for either * or $_*$ depending whether you act on covariant or contravariant objects.

⑦ A handy special case:

The Lie Bracket:

since

$$\mathcal{L}_X\left(\frac{\partial}{\partial x^\nu}\right) = -\frac{\partial X^\mu}{\partial x^\nu}\frac{\partial}{\partial x^\mu}$$

it follows that

$$\mathcal{L}_X(Y) = X^\mu \frac{\partial Y^\nu}{\partial x^\mu}\frac{\partial}{\partial x^\nu}$$

a vector

$$- Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\frac{\partial}{\partial x^\mu}$$

$$= \left[X^\mu \frac{\partial}{\partial x^\mu}, Y^\nu \frac{\partial}{\partial x^\nu}\right]$$

$[A,B] \equiv AB - BA$
Here we compute the "commutator" of linear differential operators

i.e. $\boxed{\mathcal{L}_X(Y) = [X, Y]}$

"The Lie Bracket"

(8) (Optional, i.e. don't hand it in!) exercise:

Define

(1) $\iota_X(\omega) = \omega(X) = X^\mu \omega_\mu$

vector one-form

(2) $\iota_X\, \omega \wedge \Omega = \iota_X(\omega) \wedge \Omega - \omega \wedge \iota_X(\Omega)$

1-form k-form

etc...

Show

$$\mathcal{L}_X \Omega = \iota_X(d\Omega) + d(\iota_X \Omega)$$

k-form