Homework 3

Q 1 Hamiltonian flows are symplectomorphisms (canonical transformations). I.e., they preserve the symplectic form:

$$\begin{aligned}\mathcal{L}_X \omega &= \iota_X d\omega + d(\iota_X \omega) \\ &= 2d(\omega(X, \cdot)) \\ &= 2d(dH(\cdot)) \\ &= 0.\end{aligned}$$

Explain! (Hint, feel free to write this out in component notation. $\mathcal{L}_X$ is the Lie derivative. X is the vector field corresponding to the Hamiltonian H. ι_X denotes contraction with X).

Also, you may need some background reading to understand the terms used.

Q2 Maxwell and Monopoles

The magnetic field is a two form on $\mathbb{R}^3$ and obeys the Maxwell equation

$$dB = 0 \quad \text{——} \quad (*)$$

i) Explain why setting

$$B = dA$$

solves $(*)$ trivially.

ii) Verify that

$$B = \frac{q}{2r^3}(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)$$

is locally exact by computing dA for

$$A = \frac{q(z-r)}{2r}(x\,dy - y\,dx)$$

(Here $r = \sqrt{x^2+y^2+z^2}$.)

iii) Integrate B over a two sphere to compute its first chern class

$$c_1 = -\int_{S^2} \frac{B}{2\pi}$$

The result is called the "magnetic charge".

References

P. Dirac, Proc. Roy. Soc. London. A133, 60 (1931)

T. T. Wu & C. N. Yang, Phys. Rev. D12, 3845 (1975)

T. Eguchi, P. Gilkey & A. Hanson, Phys. Rep. 66, 213 (1980).