Homework # 5

Co-adjoint orbits

In this problem we will construct the minimal representation of $Sl(3,\mathbb{R}) \equiv G$ using the orbit method.

① The Lie algebra of $G = Sl(3,\mathbb{R})$ is

$$\mathfrak{g} \equiv sl(3,\mathbb{R}) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix} \right\}$$ i.e. trace-free 3×3 real matrices.

Let $e \in \mathfrak{g}$. Check that the mapping

$$e \xmapsto[g\in G]{} g^{-1} e g$$

is a group action. (It is called the adjoint action and is equivalent to a coadjoint one because there is a natural metric – the "Killing form" – on $\mathfrak{g}$).

② Define

$$j = \begin{pmatrix} & & \\ & & \\ 1 & & \end{pmatrix}$$

verify that

$$P = \mathrm{stab}(j) = \left\{ \begin{pmatrix} a & & \\ b & \frac{1}{a^2} & \\ c & d & a \end{pmatrix} \right\}$$

is the stabilizer of j.

③ What are the dimensions of the manifolds G and P?

④ Our next aim is to study the orbit of the adjoint action

$$\mathcal{O} = \{ h^{-1} j h \mid h \in G \} \subset \mathfrak{g}$$

Explain why we may view $\mathcal{O}$ as the set of equivalence classes

$$\mathcal{O} = {}_{P}\backslash G = \{ [g] \mid g \in G, \ [hg] = [g] \text{ when } h \in P \}$$

⑤ Gel'fand & Kirillov showed $\mathcal{O}$ was a symplectic manifold with symplectic form

$$\omega = d(\operatorname{tr} j\, de\, e^{-1}) \quad e \in {}_P\backslash G$$

What is the dimension of $\mathcal{O}$?

Compute an explicit expression for ω by stealing the results derived in class for the Heisenberg group.

(Hint: you need to show that every

$[g] \in {}_P\backslash G$ may be expressed

as $[h]$ where $h \in H = \left\{ \begin{pmatrix} a & b & c \\ & 1 & d \\ & & \frac{1}{a} \end{pmatrix} \right\}$.

⑥ Let $f : {}_P\backslash G \longrightarrow \mathbb{R}$.

Explain why the right action

$$f([g]) \xmapsto[k \in G]{} f([gk])$$

allows us to write down a representation of G.

7 The last part of the problem is to write down Hamiltonians corresponding to the right G-action on $\mathcal{O}$ given in 6. First compute the vector fields corresponding ~~to~~ the following 8 flows

(i) $[g] \longmapsto \left[g\begin{pmatrix}1 & & t\\ & 1 & \\ & & 1\end{pmatrix}\right]$

(ii) $[g] \longmapsto \left[g\begin{pmatrix}1 & t & \\ & 1 & \\ & & 1\end{pmatrix}\right]$

(iii) $[g] \longmapsto \left[g\begin{pmatrix}1 & & \\ & 1 & t\\ & & 1\end{pmatrix}\right]$

(iv) $[g] \longmapsto \left[g\begin{pmatrix}e^t & & \\ & 1 & \\ & & e^{-t}\end{pmatrix}\right]$

(v) $[g] \longmapsto \left[g\begin{pmatrix}e^t & & \\ & e^{-2t} & \\ & & e^t\end{pmatrix}\right]$

(vi) $[g] \longmapsto \left[g\begin{pmatrix}1 & & \\ & 1 & \\ & t & 1\end{pmatrix}\right]$

(vii) $[g] \longmapsto \left[g\begin{pmatrix}1 & & \\ t & 1 & \\ & & 1\end{pmatrix}\right]$

(viii) $[g] \longmapsto \left[g\begin{pmatrix}1 & & \\ & 1 & \\ t & & 1\end{pmatrix}\right]$

Hint: do not hesitate to use a computer to help with the matrix algebra.

⑧ Use the symplectic form to compute the corresponding 8 Hamiltonians.

⑨ Study the algebra obeyed by the Poisson brackets of your Hamiltonians. Compare your results with the algebra obeyed by the following matrices

$$\begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix}, \begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix}, \begin{pmatrix} & & \\ & & 1 \\ & & \end{pmatrix}, \begin{pmatrix} 1 & & \\ & & \\ & & -1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & -2 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & & \\ 1 & & \\ & & \end{pmatrix},$$

$$\begin{pmatrix} & & \\ & & \\ & 1 & \end{pmatrix}, \begin{pmatrix} & & \\ & & \\ 1 & & \end{pmatrix},$$

with respect to the matrix commutator

$$[A,B] = AB - BA.$$

Comment.

(10) Study what happens to your Hamiltonians under the symplectomorphism

$$\hbar = \frac{R^2}{2}, \quad q = \frac{Rx}{2}$$

$$\pi = \frac{p}{R} - \frac{x\Pi}{R^2}, \quad p = \frac{2\Pi}{R}$$

(11) Quantize your results by setting

$$p = -i\frac{\partial}{\partial x}$$

$$\Pi = -i\frac{\partial}{\partial R}.$$

Does the algebra of Hamiltonians still work out?

If so you have successfully computed the minimal co-adjoint orbit of $Sl(3,\mathbb{R})$ with Gel'fand—Kirillov dimension 2.