Homework 8

① Queen Isabella suggests to Christopher Columbus that he might best reach the Indies by employing quantum fluctuations. She gives him enough gold to prepare his fleet in the ground state or any of the first excited states of the Hamiltonian corresponding to classical geodesic motion on the sphere. Compute these states using methods similar to those described in class for the harmonic oscillator. Explain which of the four possible states is his best choice, assuming he also diagonalizes the momentum in the direction of San Salvador Island.

② The $so(2,1) \cong su(1,1) \cong sl(2) \cong sp(2)$ Lie algebra is

$$[J_t, J_x] = i J_y$$

$$[J_t, J_y] = -i J_x$$

$$[J_x, J_y] = -i J_t$$

(i) Verify that the Casimir

$$C = J_t^2 - J_x^2 - J_y^2$$

is central. I.e. it commutes with J_t, J_x, J_y.

(ii) Label eigenstates of J_t and C by $|j\ m\rangle$ where

$$J_t |j\ m\rangle = m |j\ m\rangle$$

$$C |j\ m\rangle = j(j+1) |j\ m\rangle$$

Verify $J_\pm |j\ m\rangle = \sqrt{m(m\pm 1) - j(j+1)}\ |j\ m\pm 1\rangle$

where $J_\pm = J_x \pm i J_y$

(iii) Suppose J_t, J_x, J_y are observables and

$$\langle j\, m \,|\, j'\, m' \rangle = \delta_{jj'}\, \delta_{mm'},$$

explain why

$$m(m+1) - j(j+1) \geq 0.$$

(iv) Analyse this inequality in terms of $s = j + \frac{1}{2}$ and the allowed values of m at fixed s.

Hint s may be complex. The correct result classifies the unitary representations of $SL(2, \mathbb{R})$.