265 Take Home 1

Due Monday November 6

Question 1
The hydrogen atom is well described by the Hamiltonian

$$H = -\frac{\hbar^2}{2}\Delta - \frac{e^2}{|\vec{x}|}\,.$$

Show that the hydrogen atom is stable.

Hints

- A good strategy for demonstrating stability is to establish a lower bound on the spectrum of H (*i.e.* a minimal energy below which the system cannot decay).
- Think about an uncertainty principle for the operators $|\vec{x}|^{-1}\vec{\nabla}\ |\vec{x}|^{-1}$ and $\vec{x}$.

Question 2
Compute infinitesimal generators of the G-action on the minimal coadjoint orbit for $G = Sl(3,\mathbb{R})$. Quantize and interpret this system[1].

Question 3
Study self-adjoint extensions of $H = -\frac{\hbar^2}{2}\frac{d^2}{dx^2}$ on $L^2(0,1)$. Compute the spectrum of H in each case you find. Can you use the uncertainty principle to estimate the lowest energy eigenvalue?

Question 4
Consider the pairs of operators $(x, -i\hbar\frac{d}{dx})$ and $(S_x = \frac{1}{2}\sigma_x, S_y = \frac{1}{2}\sigma_y)$. Propose minimal uncertainty states in each case (*i.e.*, states minimizing the uncertainty bound).

[1] Hint: In the conventions employed in class you should find something like $e_{-\omega} = \hbar\pi^2 + pq\pi$. Study the symplectomorphism $\hbar = R^2/2$, $q = Rx/2$, $\pi = P/R - x\Pi/R^2$ and $p = 2\Pi/R$.

Question 5

Let $L^2(\mathbb{Z}/n\mathbb{Z}) = \{\psi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}\}$ with inner product

$$\langle \phi | \psi \rangle = \sum_{x \in \mathbb{Z}/n\mathbb{Z}} \phi^*(x)\psi(x).$$

Define the discrete Fourier transform

$$\widetilde{\psi}(x) = \sum_{y \in \mathbb{Z}/n\mathbb{Z}} \psi(y) \exp\Big(- 2\pi i y x/n \Big) \, .$$

Try to generalize as many properties of the usual Fourier transform as possible. Derive an uncertainty bound on the product of the cardinalities of the supports of ψ and $\widetilde{\psi}$.

Question 6

Ehrenfest's Theorem Consider $H = -\frac{\hbar^2}{2}\Delta + V(\vec{x})$. Show that expectations obey the classical field equation

$$\frac{d^2}{dt^2}\langle \vec{x} \rangle = -\langle \vec{\nabla} V \rangle \, .$$

Question 7

A high school has 1000 students and each has a numbered locker where they keep various smelly items. Fortunately all the locker doors are shut. One by one, each student walks past the lockers, and either opens or shuts (depending on its previous position) the door of any locker that their own locker number divides. How many lockers are open at the end?