Coordinate Vector Worksheet

So far we have dealt with abstract vectors, but many of you have already encountered "column vectors" in previous coursework. How are these related? What is going on here?? This worksheet will try explore the relationship between these notions.

First try out some warm-up questions:

(1) *What does the equation* $V = \operatorname{span}(v_1, \ldots, v_n)$ *mean?*

(2) *Define the term "finite dimensional vector space".*

(3) *Suppose* $V = \operatorname{span}(v_1, \ldots, v_n)$. *What additional requirement is necessary for* $(v_1, \ldots, v_n)$ *to be a basis for* V*?*

(4) *Define the dimension of a finite dimensional vector space.*

Suppose $v \in V$ and $(v_1, \ldots, v_n)$ is a basis for V. Moreover let

$$v = \alpha_1.v_1 + \cdots + \alpha_n.v_n\,.$$

Now answer the following crucial questions: what can you say about the numbers α_i*? Why?*

Next we need a new notation for linear combinations:

$$\alpha_1.v_1 + \alpha_2 v_2 \cdots + \alpha_n.v_n := \big(v_1, v_2, \ldots, v_n\big) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}\,.$$

We call $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ a column vector, or sometimes the coordinate vector of v. To be really precise we should say the "coordinate vector of v in the basis $(v_1, v_2, \ldots, v_n)$".

Explain why $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ *completely determines* v.

Column vectors themselves actually form a vector space. Let

$$v = \alpha_1.v_1 + \cdots + \alpha_n.v_n \qquad \& \qquad w = \beta_1.v_1 + \cdots + \beta_n.v_n\,.$$

Write down the column vectors corresponding to v, w, $v + w$ *and* $\lambda.v$. *Propose rules for vector addition and scalar multiplication of column vectors that makes the set of column vectors into a vector space.*

Now lets do an example. Let $V = \mathbb{F}_3[z]$ be the space of polynomials over $\mathbb{F}$ of degree 3 or less. This space has dimension 4 and a basis is $(1+z, z+z^2, z^2+z^3, z^3-1)$ (assume $\mathbb{F} \neq \mathbb{Z}_2$). *Write the column vectors of the following vectors in this basis*

(1) $1 + z + z^2 + z^3$
(2) 1
(3) z
(4) z^2
(5) z^3
(6) $1 - z + z^2 - z^3$

Now repeat the last exercise but in the basis $(1, z, z^2, z^3)$.

If you calculated correctly you will have found different column vectors representing the same vector. This is because we changed the basis. NOTICE THAT THE VECTORS ARE NOT CHANGING, ONLY THE WAY WE TRY TO REPRESENT THEM DOES! To go from one basis to another, a common trick is to find a "change of basis matrix". *I.e.,* a matrix that multiplies column vectors in one basis and converts them into column vectors in another basis. Now try these last few exercises:

(1) *Find out how to multiply column vectors by square matrices to produce column vectors.*
(2) *Find the square matrix that converts column vectors in the basis* $(1 + z, z + z^2, z^2 + z^3, z^3 - 1)$ *to column vectors in the basis* $(1, z, z^2, z^3)$.
(3) *Find the square matrix that converts column vectors in the basis* $(1, z, z^2, z^3)$ *to column vectors in the basis* $(1 + z, z + z^2, z^2 + z^3, z^3 - 1)$.
(4) *Find out how to multiply square matrices by one another.*
(5) *Compute the product of the two matrices you found in 2) and 3) above.*
(6) *The result of your matrix multiplication in part 5) should be the identity matrix. Find out what the identity matrix is and what its job is.*