WARNING TO GRADERS: CONUNDRUM'S SOLUTIONS ARE NOT UNIQUE, NOR OPTIMAL. GIVE FULL POINTS TO ANY WELL LAID OUT, LOGICAL & CORRECT ARGUMENT

Last Initial C

FULL Name Conundrum

Student ID π

67 MIDTERM I

3:10pm-4:00pm, Wednesday February 1, 2012

Declaration of honesty: I, the undersigned, do hereby swear to uphold the highest standards of academic honesty, including, but not limited to, submitting work that is original, my own and unaided by books, calculators, pet rocks, the secret service, computers, mobile phones, immobile phones, muses, ruses, calculus superheros or any other electronic device.

Only carefully set out work is guaranteed full credit. Focus on doing questions well rather than scoring partial credit.

Signature Cl **Date** TODAY

1	
2	
3	
4	
Σ	

Question 1

Let V be a vector space over $\mathbb{F}$. **Give definitions** for the following:

(i) A subspace U of V.

$U \subset V$, U is a vector space over $\mathbb{F}$ w.r.t. $+_V$, $\circ_V$.

(ii) $\text{span}(v_1, \ldots, v_n)$ where $v_i \in V$ $(i = 1, \ldots, n)$.

$$\text{span}(v_1, .., v_n) := \left\{ \sum_{i=1}^{n} \alpha_i v_i \,/\, \alpha_i \in \mathbb{F} \right\}$$

Now let $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ be vectors in V. Moreover suppose

$$w_1, \ldots, w_m \in \text{span}(v_1, \ldots, v_n) \qquad \text{and} \qquad v_1, \ldots, v_n \in \text{span}(w_1, \ldots, w_m).$$

Prove

$$\text{span}(v_1, \ldots, v_n) = \text{span}(w_1, \ldots, w_m).$$

Call $\text{span}(v_1, .., v_n) =: V$, $\text{span}(w_1, .., w_m) =: W$

Then need to show $v \in V \iff v \in W$

$v \in V \Rightarrow v = \sum_{i=1}^{n} \alpha_i v_i$ But $v_i \in W$ so $v_i = \sum_{j=1}^{m} \beta_j^i \cdot w_j$

Hence

$$v = \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{m} \beta_j^i w_j \right) = \sum_{j=1}^{m} \underbrace{\left(\sum_{i=1}^{n} \alpha_i \beta_j^i \right)}_{=: \tilde{\beta}_j} w_j = \sum_{j=1}^{m} \tilde{\beta}_j \cdot w_j$$

so $v \in \text{span}(w_1, .., w_m)$.

Thus $v \in V \Rightarrow v \in W$. The same argument shows $v \in W \Rightarrow v \in V$ QED

Question 2

$V^{\odot 2} = V \odot V$ means symmetric product, else add $+_V$ commutative

Give the definition of a vector space V over a field $\mathbb{F}$ (*do not define what a field is*):

$(V, +_V, \mathbb{F}, \cdot_V)$ such that $+_V : V^{\odot 2} \to V$, $\cdot_V : \mathbb{F} \times V \to V$

subject to
0) $\exists\, 0 \in V$ s.t. $v + 0 = v \;\; \forall v \in V$
i) $\forall v \in V \;\exists\, z \in V$ s.t. $v + z = 0$
ii) $1 \cdot v = v \;\; \forall v \in V$
iv) $(\alpha + \beta) \cdot v = \alpha v + \beta v$, $\alpha \cdot (v + v') = \alpha \cdot v' + \alpha \cdot v$
$\forall \alpha, \beta \in \mathbb{F}$, $v, v' \in V$
v) $(\alpha\beta) \cdot v = \alpha \cdot (\beta \cdot v)$, $u + (v + w) = (u + v) + w$
$\forall v, u, w \in V$, $\alpha, \beta \in \mathbb{F}$

Prove or disprove:

(i) The set of integers $\mathbb{Z}$ is a vector space over the rationals $\mathbb{Q}$ with standard rules for addition and scalar multiplication.

FALSE $\quad \underset{\in \mathbb{Q}}{\frac{1}{2}} \cdot \underset{\in \mathbb{Z}}{1} = \frac{1}{2} \notin \mathbb{Z}$.

(ii) The set of integers $\mathbb{Z}$ is a vector space over bits $\mathbb{Z}_2$ with standard rules for addition and scalar multiplication.

FALSE $\quad \underbrace{(1+1)}_{\in \mathbb{Z}_2} \cdot \underbrace{1}_{\in \mathbb{Z}} = 0 \cdot 1 = 0$

but $(1+1) \cdot 1 = 1 \cdot 1 + 1 \cdot 1 = 1 + 1 = 2$

so distributivity fails.

...to be continued

(iii) The subset $\{(x, y, z, w) \mid x + 2y - z + w = 0\}$ of the vector space $\mathbb{F}^4$ over $\mathbb{F}$ with addition and scalar multiplication inherited from the standard rules in $\mathbb{F}^4$ is itself a vector space.

$f : \mathbb{F}^4 \longrightarrow \mathbb{F}$

$(x, y, z, w) \mapsto x + 2y - z + w$

is linear and the above set is the kernel of this map, so certainly a subspace.

(iv) The subset $\{(x, x+y, x-y, y) \mid x, y \in \mathbb{Z}_2\}$ of the vector space $\mathbb{Z}_2^4$ over $\mathbb{Z}_2 = \{0, 1\}$ with addition and scalar multiplication inherited from the standard rules in $\mathbb{Z}_2^4$ is itself a vector space.

$f : \mathbb{Z}_2^2 \longrightarrow \mathbb{Z}_2^4$

$(x, y) \mapsto (x, x+y, x-y, y)$

is linear and the above set is $\mathrm{Im}(f)$ so certainly a subspace.

Question 3

On the next page, points in the vector space $\mathbb{Z}_3^4 = \{(x, y, z, w) | \; x, y, z, w \in \mathbb{Z}_3\}$ over $\mathbb{Z}_3 = \{0, 1, 2\}$ are depicted as positions in a 4-dimensional game of TIC-TAC-TOE. (The zero vector $(0, 0, 0, 0)$ is at the extreme top left and the vector $(2, 2, 2, 2)$ is at the bottom right.)

(i) Mark on the diagram the vectors

$$(1, 1, 0, 0)\,,\ (0, 0, 2, 2)\,,\ (1, 0, 1, 1) \text{ and } (2, 1, 0, 2)\,.$$

(Hint: To keep things tidy, use the shorthand 1100 and write small!)

(ii) Let $U = \{(x, y, z, z) | \; x, y, z \in \mathbb{Z}_3\}$. Put an O on all locations corresponding to points in U. How many O's were required? Is U a subspace?

3^3, YES

(iii) Let $V = \{(x, y, z, w) \in \mathbb{Z}_3^4 | \; x + y + z + w = 0, z + w = 0\}$. Put an X on all locations corresponding to points in V. How many X's did you need? Is V a subspace?

3^2, YES

(iv) Does

$$U + V = U \oplus V\,?$$

NO

Explain.

$U \cap V \neq \emptyset$ (there are 3 ⊗'s)

(v) Write down in set notation a subspace corresponding to a line L in $\mathbb{Z}_3^4$ such that $L + U = \mathbb{Z}_3^4$. Mark a □ for each point on your chosen line. How many squares did this take?

$L = \{(0, 0, 0, w) | \, w \in \mathbb{Z}_3\}$, 3 squares

(vi) Suppose W is a (non-empty) subspace of $\mathbb{Z}_3^4$. Let

$$w := \log_3 \big(\text{card } W\big)\,.$$

(Remember that the cardinality is the number of elements of a set.) List all possible values for w.

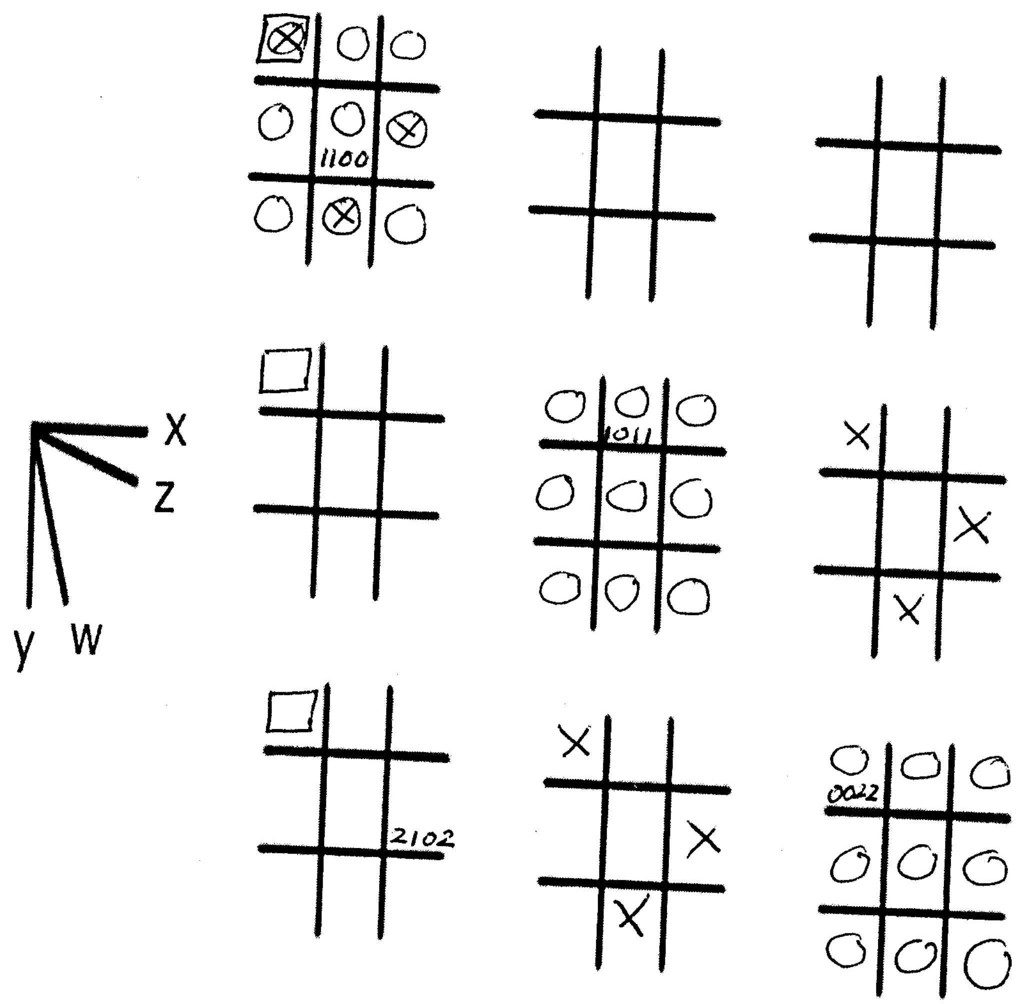
point $w = 0$

line $w = 1$ ← see part (v)

plane $w = 2$ ← see part (iii)

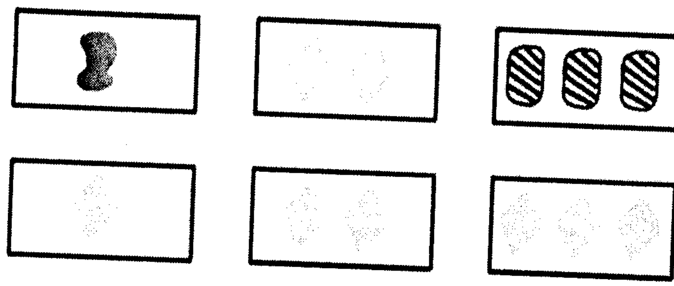
hyperplane $w = 3$ ← see part (ii)

$\mathbb{Z}_3^4$ $w = 4$



To be continued

(vii) *Harder:* In the game Set®, each card has four attributes–number, shape, shading and color for each of which there are three possibilities. Players race one to yell out "set" when they recognize three cards that, for each



$\{(x,0,0,0) \mid x \in \mathbb{F}\}$
a line through 0

Explain how to identify set cards with points in $\mathbb{Z}_3^4$. Can you write down subspaces corresponding to "sets"? — YES – SOMETIMES

$\{\text{Set Cards}\} = \{(\text{number}, \text{shape}, \text{shading}, \text{color}) \mid$
number $\in \{1,2,3\}$, shape $\in \{$peanut, diamond, oval$\}$,
shading $\in \{$full, hatched, empty$\}$,
color $\in \{$red, green, blue$\}\}$

Call $(1,2,3) \equiv (0,1,2)$ — number $= x$
(say) $(p,d,o) \equiv (1,0,2)$ — shape $= y$
$(f,h,e) \equiv (0,1,2)$ — shading $= z$
$(r,g,b) \equiv (1,0,2)$ — color $= w$

Now the above set is exactly $\mathbb{Z}_3^4$.

The question of subspaces and sets is a little tricky, clearly we care about lines ~~through the origin, through the origin~~ but subspaces also must include the origin.

A detailed explanation is at
www.rose-hulman.edu/mathjournal/archives/2007/vol8-n1/paper10/v8n1-10pd.pdf

Question 4 *(Extra Credit)*

Use this space for feedback (positive or negative) that could help improve the course:

Teach Conundrum to write neater!