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FULL Name CAPTAIN CONUNDRUM

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67 MIDTERM II

3:10pm-4:00pm, Wednesday February 19, 2012

Declaration of honesty: I, the undersigned, do hereby swear to uphold the highest standards of academic honesty, including, but not limited to, submitting work that is original, my own and unaided by books, calculators, pet rocks, the secret service, computers, mobile phones, immobile phones, muses or any other electronic device.

Only carefully set out work is guaranteed full credit. Focus on doing questions well rather than scoring partial credit.

Signature C☺C **Date** TODAY

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Question 1

Definitions:

Let V be a vector space over $\mathbb{F}$. Define the following:

(i) The vectors $v_1, \ldots v_n$ are linearly independent.

The only sol'n to $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$
is $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

(ii) The vectors $(v_1, \ldots, v_n)$ are a basis for V.

$v_1, \ldots, v_n$ are linearly independent and
$V = \text{span}(v_1, \ldots, v_n) = \{\alpha_1 v_1 + \cdots + \alpha_n v_n \mid \alpha_i \in \mathbb{F}\}$

Counting bit bases:

Consider the vector space $V = \mathbb{Z}_2^n$ over $\mathbb{Z}_2$. In the following include a *brief* justification for each answer.

(a) How many vectors are there in $\mathbb{Z}_2^n$?

2^n b/c $(\underset{\substack{\uparrow \\ 2 \\ \text{choices}}}{\ }, \overset{\substack{2 \text{ choices} \\ \downarrow}}{\ }, \ldots, \underset{\substack{\uparrow \\ 2 \\ \text{choices}}}{\ })$, 2 choices/slot, n slots

(b) Which element of $\mathbb{Z}_2^n$ is never a basis vector?

0 vector

(c) Suppose you are given a basis for $\mathbb{Z}_2^n$ whose first basis vector is f_1, how many choices remain for the second basis vector f_2?

$2^n - \underset{\substack{\uparrow \\ \text{can't use } 0 \text{ or } f_1}}{2}$

To be continued...

(d) Suppose your basis now includes f_1 and f_2. How many choices are there for the third basis vector f_3?

$$2^n - 4$$ cant use $0, f_1, f_2, f_1+f_2$

(e) Suppose your basis now includes f_1, f_2 and f_3. How many choices are there for the fourth basis vector f_4?

$$2^n - 2^3$$ can't use 0, f_1, f_2, f_3, $f_1+f_2, f_2+f_3, f_3+f_1$, $f_1+f_2+f_3$

(f) How many (ordered) bases are there for $\mathbb{Z}_2^n$?

$$(2^n-1) \times (2^n-2) \times (2^n-2^2) \times \dots \times (2^n-2^{n-1})$$ ← This suffices

$$= (2^n-1)\ 2\cdot(2^{n-1}-1) \times 2^2(2^{n-2}-1) \times \dots \times 2^{n-1}(2-1)$$

not necc.

$$= 2^{\frac{(n-1)n}{2}} (2^n-1)(2^{n-1}-1) \cdots (\cancel{2-1})$$ 1

$$= 2^{\frac{(n-1)n}{2}} \prod_{i=1}^{n} (2^i-1)$$

Question 2

Definition:

Define what it means for a ~~linear~~ map to be injective:

$f(u) = f(v) \Rightarrow u = v.$

Application:

Let V and W be vector spaces over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V, W)$ is injective. Given linearly independent vectors $v_1, \ldots, v_n$ in V, **prove** or **disprove** that the vectors $T(v_1), \ldots, T(v_n)$ are linearly independent in W.

Proof By contradiction, assume

$$\alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) = 0$$

$$\underline{\text{not all } \alpha_i = 0}$$

$$\underset{T \text{ linear}}{\Rightarrow} \quad T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = 0 = T(0)$$

$$\underset{T \text{ injective}}{\Rightarrow} \quad \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

$\Rightarrow$ $v_1, \ldots, v_n$ linearly dependent (b/c not all $\alpha_i = 0$)

$\lightning$

QED

Question 3

Definition:

Define what it means for two vector spaces to be isomorphic.

$\exists\, f : U \longrightarrow V$ $\quad$ f bijection & f linear.

Application:

Let $\mathbb{R}_+ := (0, \infty)$ and define

$$\oplus : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+ \qquad \text{and} \qquad \odot : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+ ,$$

by

$$u \oplus v = uv \qquad \text{and} \qquad \lambda \odot v = v^{\lambda} .$$

(i) Show $(\mathbb{R}_+, \oplus, \mathbb{R}, \odot)$ is a vector space.

$\oplus$ closure: product of positive numbers is positive

$\odot$ closure: positive number to power of positive number is positive

$\oplus$ commutes: $uv = vu$

$\oplus$ identity: zero vector is 1 b/c $1 \cdot u = u = u \cdot 1$

$\odot$ identity: $u^1 = u$

$\oplus, \odot$ associate: $u(vw) = (uv)w$, $(u^{\lambda})^{\mu} = u^{\lambda\mu} = u^{(\lambda\mu)}$

$\oplus, \odot$ distribute: $u^{(\lambda+\mu)} = (u^{\lambda})(u^{\mu})$, $(uv)^{\lambda} = u^{\lambda}v^{\lambda}$

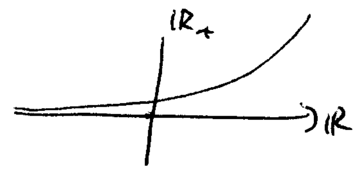
$\oplus$ inverse: inverse of u is $\frac{1}{u}$ b/c $u \frac{1}{u} = 1$

To be continued...

(ii) Prove that the vector spaces $(\mathbb{R}_+, \oplus, \mathbb{R}, \otimes)$ and $(\mathbb{R}, +, \mathbb{R}, .)$ are isomorphic. *Hint: Try a constructive proof by choosing a special invertible function* $\mathbb{R} \to \mathbb{R}_+$.

Let

$$\begin{aligned} T : \mathbb{R} &\longrightarrow \mathbb{R}_+ \\ v &\longmapsto e^{v} \end{aligned}$$



st quired prove invertiblity

The function e^{v} is monotone increasing and $e^{\mathbb{R}} = \mathbb{R}_+$ so T is a bijection (its inverse is $\ln$).

Linearity:

$$T(\lambda . u + \mu . v)$$

$$= \exp(\lambda u + \mu v)$$

$$= e^{\lambda u} e^{\mu v}$$

$$= (e^{u})^{\lambda} (e^{v})^{\mu}$$

$$= (T(u))^{\lambda} (T(v))^{\mu}$$

$$= (\lambda \odot T(u)) \oplus (\mu \odot T(v))$$ LINEAR

QED